# UNIT 8
# Z-Transforms – 2

## 8.1 Transform analysis of LTI systems:

- We have defined the transfer function as the $z$-transform of the impulse response of an LTI system

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

- Then we have $y[n] = x[n] * h[n]$ and $Y(z) = X(z)H(z)$
- This is another method of representing the system
- The transfer function can be written as

$$H(z) = \frac{Y(z)}{X(z)}$$

- This is true for all $z$ in the ROCs of $X(z)$ and $Y(z)$ for which $X(z)$ in nonzero
- The impulse response is the $z$-transform of the transfer function
- We need to know ROC in order to uniquely find the impulse response
- If ROC is unknown, then we must know other characteristics such as stability or causality in order to uniquely find the impulse response

**System identification**

- Finding a system description by using input and output is known as system identification
- Ex1: find the system, if the input is $x[n] = (-1//3)^n u[n]$ and the out is $y[n] = 3(-1)^n u[n] + (1/3)^n u[n]$

- Solution: Find the $z$-transform of input and output. Use $X(z)$ and $Y(z)$ to find $H(z)$, then find $h(n)$ using the inverse $z$-transform

$$X(z) = \frac{1}{(1+(\frac{1}{3})z^{-1})}, \qquad \text{with ROC } |z| > \frac{1}{3}$$

$$Y(z) = \frac{3}{(1+z^{-1})} + \frac{1}{(1-(\frac{1}{3})z^{-1})}, \quad \text{with ROC } |z| > 1$$

- We can write $Y(z)$ as

$$Y(z) = \frac{4}{(1+z^{-1})(1-(\frac{1}{3})z^{-1})}, \quad \text{with ROC } |z| > 1$$

- We know $H(z) = Y(z)/X(z)$, so we get

$$H(z) = \frac{4(1+(\frac{1}{3})z^{-1})}{(1+z^{-1})(1-(\frac{1}{3})z^{-1})} \quad \text{with ROC } |z| > 1$$

- We need to find inverse $z$-transform to find $x[n]$, so use partial fraction and write $H(z)$ as

$$H(z) = \frac{2}{1+z^{-1}} + \frac{2}{1-(\frac{1}{3})z^{-1}} \quad \text{with ROC } |z| > 1$$

- Impulse response $x[n]$ is given by

$$h[n] = 2(-1)^n u[n] + 2(1/3)^n u[n]$$

**Relation between transfer function and difference equation**

- The transfer can be obtained directly from the difference-equation description of an LTI system

- We know that

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- We know that the transfer function $H(z)$ is an eigen value of the system associated with the eigen function $z^n$, ie. if $x[n] = z^n$ then the output of an LTI system $y[n] = z^n H(z)$

- Put $x[n-k] = z^{n-k}$ and $y[n-k] = z^{n-k} H(z)$ in the difference equation, we get

$$z^n \sum_{k=0}^{N} a_k z^{-k} H(z) = z^n \sum_{k=0}^{M} b_k z^{-k}$$

- We can solve for $H(z)$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- The transfer function described by a difference equation is a ratio of polynomials in $z^{-1}$ and is termed as a rational transfer function.
- The coefficient of $z^{-k}$ in the numerator polynomial is the coefficient associated with $x[n-k]$ in the difference equation
- The coefficient of $z^{-k}$ in the denominator polynomial is the coefficient associated with $y[n-k]$ in the difference equation
- This relation allows us to find the transfer function and also find the difference equation description for a system, given a rational function

**Transfer function:**

- The poles and zeros of a rational function offer much insight into LTI system characteristics
- The transfer function can be expressed in pole-zero form by factoring the numerator and denominator polynomial
- If $c_k$ and $d_k$ are zeros and poles of the system respectively and $\tilde{b} = b_0/a_0$ is the gain factor, then

$$H(z) = \frac{\tilde{b}\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

- This form assumes there are no poles and zeros at $z = 0$
- The $p^{th}$ order pole at $z = 0$ occurs when $b_0 = b_1 = \ldots = b_{p-1} = 0$
- The $l^{th}$ order zero at $z = 0$ occurs when $a_0 = a_1 = \ldots = a_{l-1} = 0$
- Then we can write $H(z)$ as

$$H(z) = \frac{\tilde{b}z^{-p}\prod_{k=1}^{M-p}(1 - c_k z^{-1})}{z^{-l}\prod_{k=1}^{N-l}(1 - d_k z^{-1})}$$

where $\tilde{b} = b_p/a_l$

- In the example we had first order pole at $z = 0$
- The poles, zeros and gain factor $\tilde{b}$ uniquely determine the transfer function
- This is another description for input-output behavior of the system
- The poles are the roots of characteristic equation

## 8.2 Unilateral Z- transforms:

- Useful in case of causal signals and LTI systems
- The choice of time origin is arbitrary, so we may choose $n = 0$ as the time at which the input is applied and then study the response for times $n \geq 0$

### Advantages

- We do not need to use ROCs
- It allows the study of LTI systems described by the difference equation with initial conditions

### Unilateral $z$-transform

- The unilateral $z$-transform of a signal $x[n]$ is defined as

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

which depends only on $x[n]$ for $n \geq 0$

- The unilateral and bilateral $z$-transforms are equivalent for causal signals

$$\alpha^n u[n] \xleftrightarrow{z_u} \frac{1}{1 - \alpha z^{-1}}$$

$$a^n \cos(\Omega_o n) u[n] \xleftrightarrow{z_u} \frac{1 - a\cos(\Omega_o) z^{-1}}{1 - 2a\cos(\Omega_o) z^{-1} + a^2 z^{-2}}$$

## Properties of unilateral Z transform:

- Consider the difference equation description of an LTI system

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- We may write the $z$-transform as

$$A(z)Y(z) + C(z) = B(z)X(z)$$

where

$$A(z) = \sum_{k=0}^{N} a_k z^{-k} \quad \text{and} \quad B(z) = \sum_{k=0}^{M} b_k z^{-k}$$

- The same properties are satisfied by both unilateral and bilateral $z$-transforms with one exception: the time shift property

- The time shift property for unilateral $z$-transform: Let $w[n] = x[n-1]$
  - The unilateral $z$-transform of $w[n]$ is

$$W(z) = \sum_{n=0}^{\infty} w[n]z^{-n} = \sum_{n=0}^{\infty} x[n-1]z^{-n}$$

$$W(z) = x[-1] + \sum_{n=1}^{\infty} x[n-1]z^{-n}$$

$$W(z) = x[-1] + \sum_{m=0}^{\infty} x[m]z^{-(m+1)}$$

  - The unilateral $z$-transform of $w[n]$ is

$$W(z) = x[-1] + z^{-1} \sum_{m=0}^{\infty} x[m]z^{-m}$$

$$W(z) = x[-1] + z^{-1}X(z)$$

- A one-unit time shift results in multiplication by $z^{-1}$ and addition of the constant $x[-1]$

- In a similar way, the time-shift property for delays greater than unity is

$$x[n-k] \xleftrightarrow{z_u} x[-k] + x[-k+1]z^{-1} +$$
$$\ldots + x[-1]z^{-k+1} + z^{-k}X(z) \ \text{ for } \ k > 0$$

- In the case of time advance, the time-shift property changes to

$$x[n+k] \xleftrightarrow{z_u} -x[0]z^{k} - x[-1]z^{k-1} +$$
$$\ldots - x[k-1]z + z^{k}X(z) \ \text{ for } \ k > 0$$

## 8.3 Application to solve difference equations

**Solving Differential equations using initial conditions:**

- We get

$$C(z) = \sum_{m=0}^{N-1} \sum_{k=m+1}^{N} a_k y[-k+m] z^{-m}$$

- We have assumed that $x[n]$ is causal and

$$x[n-k] \stackrel{z_u}{\longleftrightarrow} z^{-k} X(z)$$

- The term $C(z)$ depends on the $N$ initial conditions $y[-1], y[-2], \ldots, y[-N]$ and the $a_k$
- $C(z)$ is zero if all the initial conditions are zero

- Solving for $Y(z)$, gives

$$Y(z) = \frac{B(z)}{A(z)} X(z) - \frac{C(z)}{A(z)}$$

- The output is the sum of the forced response due to the input and the natural response induced by the initial conditions
- The forced response due to the input

$$\frac{B(z)}{A(z)} X(z)$$

- The natural response induced by the initial conditions

$$\frac{C(z)}{A(z)}$$

- $C(z)$ is the polynomial, the poles of the natural response are the roots of $A(z)$, which are also the poles of the transfer function

- The form of natural response depends only on the poles of the system, which are the roots of the characteristic equation

**First order recursive system**

- Consider the first order system described by a difference equation

$$y[n] - \rho y[n-1] = x[n]$$

  where $\rho = 1 + r/100$, and $r$ is the interest rate per period in percent and $y[n]$ is the balance after the deposit or withdrawal of $x[n]$

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- Assume bank account has an initial balance of \$10,000/- and earns 6% interest compounded monthly. Starting in the first month of the second year, the owner withdraws \$100 per month from the account at the beginning of each month. Determine the balance at the start of each month.

- Solution: Take unilateral $z$-transform and use time-shift property we get

$$Y(z) - \rho(y[-1] + z^{-1}Y(z)) = X(z)$$

- Rearrange the terms to find $Y(z)$, we get

$$(1 - \rho z^{-1})Y(z) = X(z) + \rho y[-1]$$

$$Y(z) = \frac{X(z)}{1 - \rho z^{-1}} + \frac{\rho y[-1]}{1 - \rho z^{-1}}$$

- $Y(z)$ consists of two terms
  - one that depends on the input: the forced response of the system
  - another that depends on the initial conditions: the natural response of the system
- The initial balance of \$10,000 at the start of the first month is the initial condition $y[-1]$, and there is an offset of two between the time index $n$ and the month index
- $y[n]$ represents the balance in the account at the start of the $n+2^{nd}$ month.
- We have $\rho = 1 + \frac{\frac{6}{12}}{100} = 1.005$
- Since the owner withdraws \$100 per month at the start of month 13 $(n = 11)$
- We may express the input to the system as $x[n] = -100u[n-11]$, we get

$$X(z) = \frac{-100z^{-11}}{1-z^{-1}}$$

- We get

$$Y(z) = \frac{-100z^{-11}}{(1-z^{-1})(1-1.005z^{-1})} + \frac{1.005(10,000)}{1-1.005z^{-1}}$$

- After a partial fraction expansion we get

$$Y(z) = \frac{20,000z^{-11}}{1-z^{-1}} + \frac{20,000z^{-11}}{1-1.005z^{-1}} + \frac{10,050}{1-1.005z^{-1}}$$

- Monthly account balance is obtained by inverse $z$-transforming $Y(z)$ We get

$$y[n] = 20,000u[n-11] - 20,000(1.005)^{n-11}u[n-11]$$
$$+ 10,050(1.005)^{n}u[n]$$

- The last term $10,050(1.005)^n u[n]$ is the natural response with the initial balance
- The account balance
- The natural balance
- The forced response

## Recommended Questions

1. Find the inverse Z transform of

$$H(z) = \frac{1 + Z^{-1}}{(1 - 0.9\, e^{j\pi/4} z^{-1})(1 - 0.9\, e^{-j\pi/4} z^{-1})}$$

2. A system is described by the difference equation

$$Y(n) - yn - 1) + \frac{1}{4} y(n-2) = x(n) + 1/4\, x(n-1) - 1/8\, x(n-2)$$

Find the Transfer function of the Inverse system
Does a stable and causal Inverse system exists

3. Sketch the magnitude response for the system having transfer functions.

4. Find the z-transform of the following x[n]:
   (a) $x[n] = \{\frac{1}{2}, 1, -\frac{1}{3}\}$
   (b) $x[n] = 2\delta[n+2] - 3\delta[n-2]$
   (c) $x[n] = 3(-\frac{1}{2})^n u[n] - 2(3)^n u[-n-1]$
   (d) $x[n] = 3(\frac{1}{7})^n u[n] - 2(\frac{1}{4})^n u[-n-1]$

5. Given

$$X(z) = \frac{z(z-4)}{(z-1)(z-2)(z-3)}$$

(a) State all the possible regions of convergence.
(b) For which ROC is X (z) the z-transform of a causal sequence?

6. Show the following properties for the z-transform.
   (a) If $x[n]$ is even, then $X(z^{-1}) = X(z)$.
   (b) If $x[n]$ is odd, then $X(z^{-1}) = -X(z)$.
   (c) If $x[n]$ is odd, then there is a zero in $X(z)$ at $z = 1$.

7. Derive the following transform pairs:

$$(\cos \Omega_0 n)u[n] \longleftrightarrow \frac{z^2 - (\cos \Omega_0)z}{z^2 - (2\cos \Omega_0)z + 1} \qquad |z| > 1$$

$$(\sin \Omega_0 n)u[n] \longleftrightarrow \frac{(\sin \Omega_0)z}{z^2 - (2\cos \Omega_0)z + 1} \qquad |z| > 1$$

8. Find the z-transforms of the following x[n]:
   (a) $x[n] = (n-3)u[n-3]$
   (b) $x[n] = (n-3)u[n]$
   (c) $x[n] = u[n] - u[n-3]$
   (d) $x[n] = n\{u[n] - u[n-3]\}$

9. Using the relation

$$a^n u[n] \longleftrightarrow \frac{z}{z-a} \qquad |z| > |a|$$

find the z-transform of the following ***x[n]:***

(a) $x[n] = na^{n-1}u[n]$

(b) $x[n] = n(n-1)a^{n-2}u[n]$

(c) $x[n] = n(n-1)\cdots(n-k+1)a^{n-k}u[n]$

10. Using the z-transform

(a) $x[n] * \delta[n] = x[n]$

(b) $x[n] * \delta[n-n_0] = x[n-n_0]$

11. Find the inverse z-transform of $X(z)= e^{a/z}$ , $z > 0$

12. Using the method of long division, find the inverse z-transform of the following ***X(z):***

(a) $X(z) = \dfrac{z}{(z-1)(z-2)}, \ |z| < 1$

(b) $X(z) = \dfrac{z}{(z-1)(z-2)}, \ 1 < |z| < 2$

(c) $X(z) = \dfrac{z}{(z-1)(z-2)}, \ |z| > 2$

13. Consider the system shown in Fig. 4-9. Find the system function $H(z)$ and its impulse response h[n]

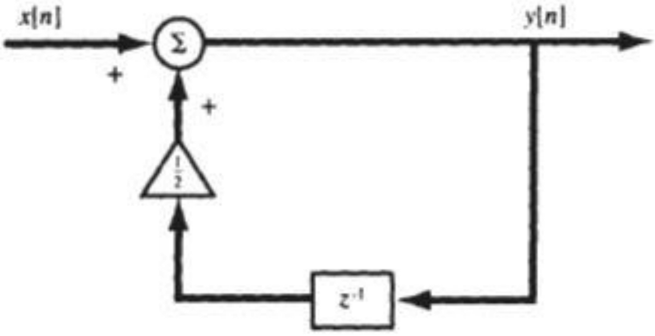


**Fig. 4-9**

14. Consider a discrete-time LTI system whose system function ***H(z)*** is given **by**

$$H(z) = \frac{z}{z-\frac{1}{2}} \qquad |z| > \frac{1}{2}$$

(a) Find the step response s[n].
(b) Find the output y[n] to the input x[n] = nu[n].

**15.** Consider a causal discrete-time system whose output y[n] and input x[n] are related by

$$y[n] - \tfrac{5}{6}y[n-1] + \tfrac{1}{6}y[n-2] = x[n]$$

(a) Find its system function H(z).
(b) Find its impulse response h[n].