Chapter 2 Network Topology

2.1 Introduction

An important step in the procedure for solving any circuit problem consists first in selecting a number of independent branch currents as (known as loop currents or mesh currents) variables, and then to express all branch currents as functions of the chosen set of branch currents. Alternately a number of independent node pair voltages may be selected as variables and then express all existing node pair voltages in terms of these selected variables.

For simple networks involving a few elements, there is no difficulty in selecting the independent branch currents or the independent node-pair voltages. The set of linearly independent equations can be written by inspection. However for large scale networks particularly modern electronic circuits such as integrated circuits and microcircuits with a larger number of interconnected branches, it is almost impossible to write a set of linearly independent equations by inspection or by mere intuition. The problem becomes quite difficult and complex. A systematic and step by step method is therefore required to deal with such networks. Network topology (graph theory approach) is used for this purpose. By this method, a set of linearly independent loop or node equations can be written in a form that is suitable for a computer solution.

2.2 Terms and definitions

The description of networks in terms of their geometry is referred to as network topology. The adequacy of a set of equations for analyzing a network is more easily determined topologically than algebraically.

Graph (or linear graph): A network graph is a network in which all nodes and loops are retained but its branches are represented by lines. The voltage sources are replaced by short circuits and current sources are replaced by open circuits. (Sources without internal impedances or admittances can also be treated in the same way because they can be shifted to other branches by E-shift and/or I-shift operations.)

Branch: A line segment replacing one or more network elements that are connected in series or parallel.

Node: Interconnection of two or more branches. It is a terminal of a branch. Usually interconnections of three or more branches are nodes.

Path: A set of branches that may be traversed in an order without passing through the same node more than once.

Loop: Any closed contour selected in a graph.

Mesh: A loop which does not contain any other loop within it.

Planar graph: A graph which may be drawn on a plane surface in such a way that no branch passes over any other branch.

Non-planar graph: Any graph which is not planar.

Oriented graph: When a direction to each branch of a graph is assigned, the resulting graph is called an oriented graph or a directed graph.

Connected graph: A graph is connected if and only if there is a path between every pair of nodes.

Sub graph: Any subset of branches of the graph.

Tree: A connected sub-graph containing all nodes of a graph but no closed path. i.e. it is a set of branches of graph which contains no loop but connects every node to every other node not necessarily directly. A number of different trees can be drawn for a given graph.

Link: A branch of the graph which does not belong to the particular tree under consideration. The links form a sub-graph not necessarily connected and is called the co-tree.

Tree compliment: Totality of links i.e. Co-tree.

Independent loop: The addition of each link to a tree, one at a time, results one closed path called an independent loop. Such a loop contains only one link and other tree branches. Obviously, the number of such independent loops equals the number of links.

Tie set: A set of branches contained in a loop such that each loop contains one link and the remainder are tree branches.

Tree branch voltages: The branch voltages may be separated in to tree branch voltages and link voltages. The tree branches connect all the nodes. Therefore if the tree branch voltages are forced to be zero, then all the node potentials become coincident and hence all branch voltages are forced to be zero. As the act of setting only the tree branch voltages to zero forces all voltages in the network to be zero, it must be possible to express all the link voltages uniquely in terms of tree branch voltages. Thus tree branch form an independent set of equations.

Cut set: A set of elements of the graph that dissociates it into two main portions of a network such that replacing any one element will destroy this property. It is a set of branches that if removed divides a connected graph in to two connected sub-graphs. Each cut set contains one tree branch and the remaining being links.

Fig. 2.1 shows a typical network with its graph, oriented graph, a tree, co-tree and a non-planar graph.

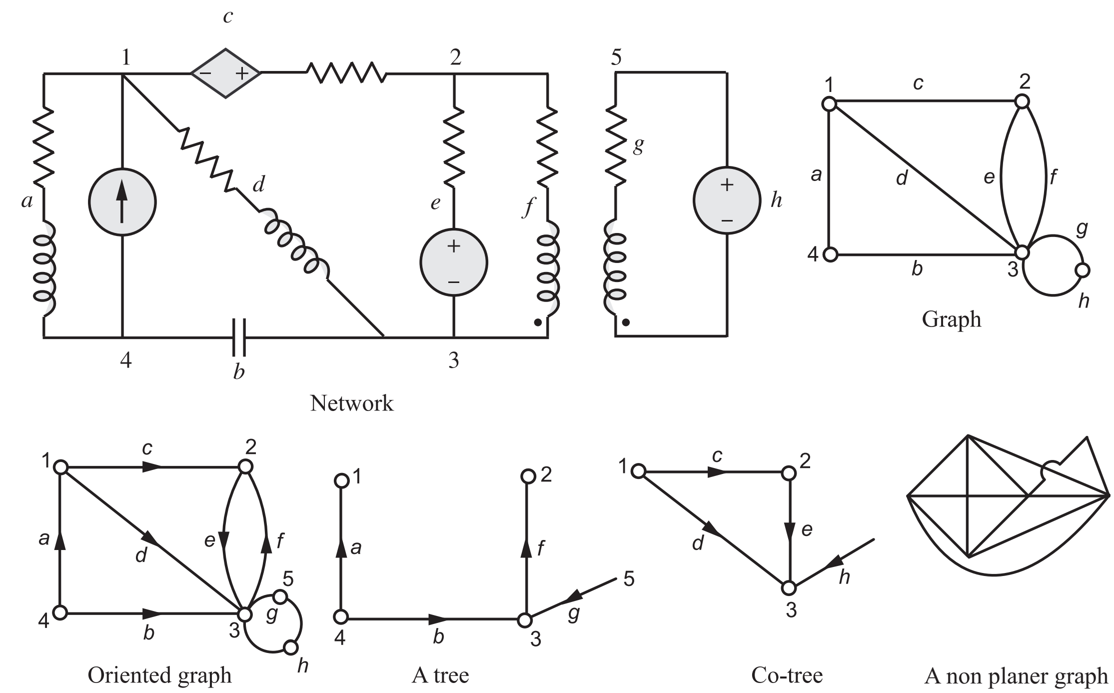

Figure 2.1

Relation between nodes, links, and branches

Let B = Total number of branches in the graph or network
N = total nodes
L = link branches

Then $N-1$ branches are required to construct a tree because the first branch chosen connects two nodes and each additional branch includes one more node.

Therefore number of independent node pair voltages = $N-1$ = number of tree branches.

Then $L = B-(N-1) = B-N+1$

Number of independent loops = $B-N+1$

2.3 Isomorphic graphs

Two graphs are said to be ismorphic if they have the same incidence matrix, though they look different. It means that they have the same numbers of nodes and the same numbers of branches. There is one to one correspondence between the nodes and one to one correspondence between the branches. Fig. 2.2 shows such graphs.

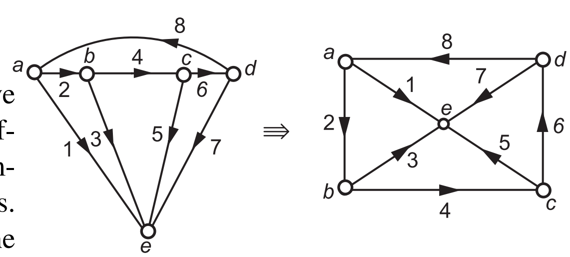

Figure 2.2

2.4 Matrix representation of a graph

For a given oriented graph, there are several representative matrices. They are extremely important in the analytical studies of a graph, particularly in the computer aided analysis and synthesis of large scale networks.

2.4.1 Incidence Matrix $\mathbf{A}_n$

It is also known as augmented incidence matrix. The element node incidence matrix **A** indicates in a connected graph, the incidence of elements to nodes. It is an $N \times B$ matrix with elements of $\mathbf{A}_n = (a_{kj})$

$$
\begin{aligned}
a_{kj} &= 1, \text{ when the branch } b_j \text{ is incident to and oriented away from the } k^{\text{th}} \; node. \\
&= -1, \text{ when the branch } b_j \text{ is incident to and oriented towards the } k^{\text{th}} \text{ node.} \\
&= 0, \text{ when the branch } b_j \text{ is not incident to the } k^{\text{th}} \text{ node.}
\end{aligned}
$$

As each branch of the graph is incident to exactly two nodes,

$$\sum_{k=0}^{n} a_{kj} = 0 \quad \text{for} \quad j = 1, 2, 3, \cdots B.$$

That is, each column of $\mathbf{A}_n$ has exactly two non zero elements, one being +1 and the other -1. Sum of elements of any column is zero. The columns of $\mathbf{A}_n$ are lineraly dependent. The rank of the matrix is less than N.

Significance of the incidence matrix lies in the fact that it translates all the geometrical features in the graph into an algebraic expression.

Using the incidence matrix, we can write KCL as

$\mathbf{A}_n \, \mathbf{i}_B = 0$, where $\mathbf{i}_B$ = branch current vector.

But these equations are not linearly independent. The rank of the matrix **A** is $N - 1$. This property of $\mathbf{A}_n$ is used to define another matrix called reduced incidence matrix or bus incidence matrix.

For the oriented graph shown in Fig. 2.3(a), the incidence matrix is as follows:

$$
\text{Nodes} \downarrow \qquad \text{branches}
$$

$$
\mathbf{A}_n = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}
\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{bmatrix} -1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}
$$

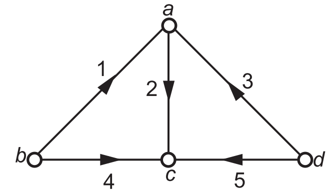

Figure 2.3(a)

Note that sum of all elements in each column is zero.

2.4.2 Reduced incidence matrix

Any node of a connected graph can be selected as a reference node. Then the voltages of the other nodes (referred to as buses) can be measured with respect to the assigned reference. The matrix obtained from $\mathbf{A}_n$ by deleting the row corresponding to the reference node is the element-bus incident matrix $\mathbf{A}$ and is called bus incidence matrix with dimension $(N-1) \times B$. $\mathbf{A}$ is rectangular and therefore singular.

In $\mathbf{A}_n$, the sum of all elements in each column is zero. This leads to an important conclusion that if one row is not known in **A**, it can be found so that sum of elements of each column must be zero.

From **A**, we have $\mathbf{A}\,\mathbf{i}_B = 0$, which represents a set of linearly independent equations and there are $N-1$ independent node equations.

For the graph shown in Fig 2.3(a), with d selected as the reference node, the reduced incidence matrix is

$$\begin{array}{cc} \text{Nodes} \downarrow & \text{branches} \\ & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \end{array}$$

$$\mathbf{A} = \begin{array}{c} a \\ b \\ c \end{array} \begin{bmatrix} -1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & -1 \end{bmatrix}$$

Note that the sum of elements of each column in **A** need not be zero.

Note that if branch current vector, $$\mathbf{j}_B = \begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \end{bmatrix}$$

Then $\mathbf{A}\,\mathbf{i}_B = 0$ representing a set of independent node equations.

Another important property of **A** is that determinant $\mathbf{A}\mathbf{A}^{\mathrm{T}}$ gives the number of possible trees of the network. If $\mathbf{A} = [\mathbf{A}_t \ : \ \mathbf{A}_i]$ where $\mathbf{A}_t$ and $\mathbf{A}_i$ are sub-matrices of **A** such that $\mathbf{A}_t$ contains only twigs, then det $\mathbf{A}_t$ is either + 1 or -1.

To verify the property that det $\mathbf{A}\mathbf{A}^{\mathrm{T}}$ gives the number of all possible trees, consider the reduced incidence matrix **A** of the example considered. That is,

$$\mathbf{A} = \left[\begin{array}{ccc:cc} 1 & 2 & 3 & 4 & 5 \\ -1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & -1 \end{array}\right], \quad \underbrace{}_{\mathbf{A}_t} \ \underbrace{}_{\mathbf{A}_i}$$

Then,

$$\text{Det } \mathbf{A}\mathbf{A}^{\mathrm{T}} = \left| \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}^{\mathrm{T}} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -1 & -1 \end{bmatrix}^{\mathrm{T}} \right| = 8$$

Fig. 2.3(b) shows all possible trees corresponding to the matrix **A**.

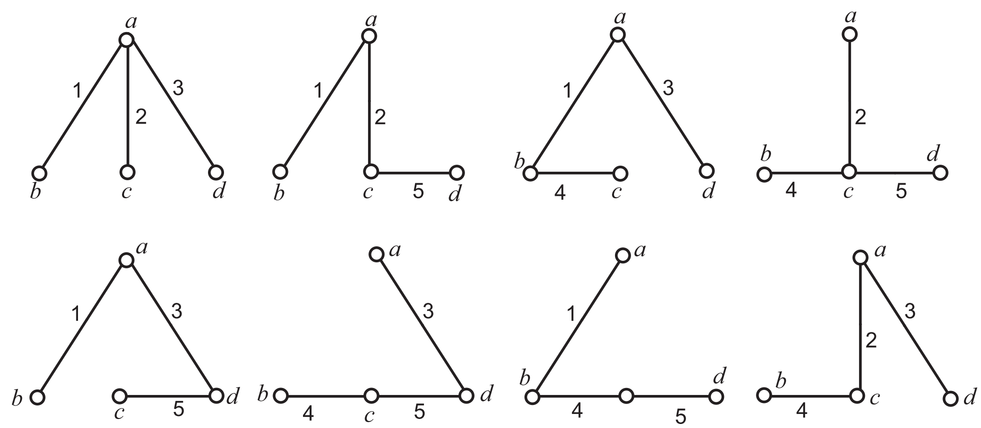

Figure 2.3(b)

To verify the property that the determinant of sub matrix $\mathbf{A}_t$ of $\mathbf{A} = \mathbf{A}_t \,;\, \mathbf{A}_i$ is +1 or -1.

For tree [2, 3, 4]

From

$$\begin{array}{c} \text{Nodes} \downarrow \qquad \text{branches} \\ \qquad\quad 2 \quad 3 \quad 4 \quad 1 \quad 5 \end{array}$$

$$\mathbf{A} = \begin{array}{c} a \\ b \\ c \end{array} \begin{bmatrix} +1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ -1 & 0 & -1 & 0 & -1 \end{bmatrix} = \mathbf{A}_t; \mathbf{A}_i$$

$$\text{Det } \mathbf{A}_i = \begin{vmatrix} 1 & +1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & -1 \end{vmatrix} = 1$$

For another tree [2, 4, 5]

$$\begin{array}{c} \text{Nodes} \downarrow \qquad \text{branches} \\ \qquad\quad 2 \quad 4 \quad 5 \quad 1 \quad 3 \end{array}$$

$$\mathbf{A} = \begin{array}{c} a \\ b \\ c \end{array} \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 \end{bmatrix} = \mathbf{A}_t; \mathbf{A}_i$$

$$\text{Det } \mathbf{A}_i = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & -1 \end{vmatrix} = -1$$

2.5 Loop equations and fundamental loop matrix (Tie-set Matrix)

From the knowledge of the basic loops (tie-sets), we can obtain loop matrix. In this matrix, the loop orientation is to be the same as the corresponding link direction. In order to construct this matrix, the following procedure is to be followed.

1. Draw the oriented graph of the network. Choose a tree.
2. Each link forms an independent loop. The direction of this loop is same as that of the corresponding link. Choose each link in turn.
3. Prepare the tie-set matrix with elements b_{iJ},

where $b_{iJ} = 1,$ when branch b_J in loop i and is directed in the same direction as the loop current.
$= -1,$ when branch b_J is in loop i and is directed in the opposite direction as the loop current.
$= 0,$ when branch b_J is not in loop i.

Tie-set matrix is an $i \times b$ matrix.
Consider the example of Fig. 2.3(a).

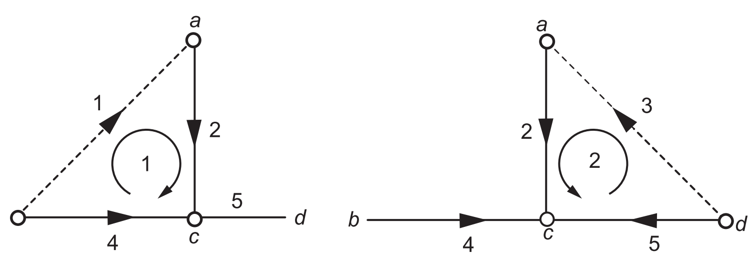

Figure 2.4

Selecting (2, 4, 5) as tree, the co-tree is (1, 3). Fig. 2.4 leads to the following tie-set.

$$\begin{array}{c} \text{Loops} \downarrow \qquad \qquad \text{branches} \\ \mathbf{M} = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \end{bmatrix} \end{array} \end{array}$$

with
$$\mathbf{V}_B = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}$$

Then $\mathbf{MV}_B$ gives the following independent loop equations:

$$v_1 + v_2 - v_4 = 0$$
$$v_2 + v_3 - v_5 = 0$$

Looking column wise, we can express branch currents in terms of loop currents. This is done by the following matrix equation.

$$\boldsymbol{J}_B = \mathbf{M}^{\mathrm{T}} \mathbf{I}_L$$

The above matrix equation gives $J_1 = i_1$, $J_2 = i_1 + i_2$, $J_3 = i_2$, $J_4 = -i_1 - i_2$
Note that J stands for branch current while i stands for loop current.
In this matrix,

(i) Each row corresponds to an independent loop. Therefore the columns of the resulting schedule automatically yield a set of equations relating each branch current to the loop currents.

(ii) As each column expresses a branch current in terms of loop currents, the rows of the matrix automatically yield the closed paths in which the associated loop currents circulate. Expressions for branch currents in terms of loop currents may be obtained in matrix form as $\mathbf{J}_B = \mathbf{M}^{\mathrm{T}}\mathbf{I}_L$.

where $\mathbf{M}$ is the tie-set matrix of $L \times B$.

In the present example,

$$J_B = \begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \end{bmatrix}$$

and

$$I_L = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

2.6 Cut-set matrix and node pair potentials

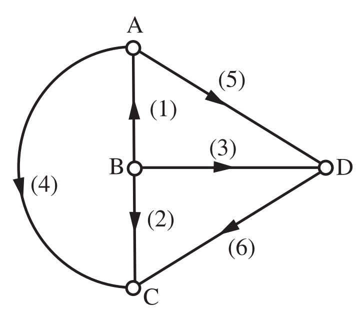

Figure 2.5(a) A directed graph

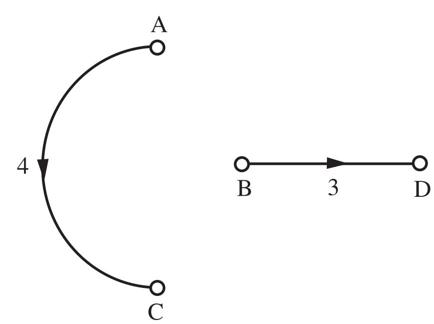

Figure 2.5(b) Two separate graphs created by the cut set {1, 2, 5, 6}

A cut-set of a graph is a set of branches whose removal, cuts the connected graph into two parts such that the replacement of any one branch of the cut-set renders the two parts connected. For example, two separated graphs are obtained for the graph of Fig. 2.5(a) by selecting the cut-set consisting of branches [1, 2, 5, 6]. These seperated graphs are as shown in Fig. 2.5(b).

Just as a systematic method exists for the selection of a set of independent loop current variables, a similar process exists for the selection of a set of independent node pair potential variables.

It is already known that the cut set is a minimal set of branches of the graph, removal of which divides the graph in to two connected sub-graphs. Then it separates the nodes of the graph in to two groups, each being one of the two sub-graphs. Each branch of the tie-set has one of its terminals incident at a node on one sub-graph. Selecting the orientation of the cut set same as that of the tree branch of the cut set, the cut set matrix is constructed row-wise taking one cut set at a time. Without link currents, the network is inactive. In the same way, without node pair voltages the network is active. This is because when one twig voltage is made active with all other twig voltages are zero, there is a set of branches which becomes active. This set is called cut-set. This set is obtained by cutting the graph by a line which cuts one twig and some links. The algebraic sum of these branch currents is zero. Making one twig voltage active in turn, we get entire set of node equations.

This matrix has current values,

$q_{iJ} = 1$, if branch J is in the cut-set with orientation same as that of tree branch.
$= -1$, if branch J is in the cut-set with orientation opposite to that of tree branch.
$= 0$, if branch J is not in the cut-set.
and dimension is $(N - 1) \times B$.

Row-by-row reading, it gives the KCL at each node and therefore we have $\mathbf{QJ}_B = 0$.

The procedure to write cut-set matrix is as follows:

(i) Draw the oriented graph of a network and choose a tree.

(ii) Each tree branch forms an independent cut-set. The direction of this cut-set is same as that of the tree branch. Choose each tree branch in turn to obtain the cut set matrix. Isolate the tree element pairs and energize each bridging tree branch. Assuming the bridging tree branch potential equals the node pair potential, thus regarding it as an independent variable.

(iii) Use the columns of the cut-set matrix to yield a set of equations relating the branch potentials in terms of the node pair potentials. This may be obtained in matrix form as $\mathbf{V}_B = \mathbf{Q}^{\mathrm{T}}\mathbf{E}_N$

where v and e are used to indicate branch potential and node voltage respectively.

In the example shown in Fig 2.5 (c), (3, 4, 5) are tree branches. Links are shown in dotted lines. If two tree branch voltages in 3 and 4 are made zero, the nodes a and c are at the same potential. Similarly the nodes b and d are at the same potential. The graph is reduced to the form shown in Fig. 2.5(d) containing only the cut-set branches. Then, we have

$$i_5 - i_1 - i_2 - i_6 = 0$$

Similarly by making only e_4 to exist (with e_5 and e_3 zero), the nodes a, b and c are at the same potential, reducing the graph to the form shown in Fig. 2.5(e). Thus,

$$i_4 + i_2 + i_6 = 0$$