2.1 Introduction

An important step in the procedure for solving any circuit problem consists first in selecting a number of independent branch currents as (known as loop currents or mesh currents) variables, and then to express all branch currents as functions of the chosen set of branch currents. Alternatively a number of independent node pair voltages may be selected as variables and then express all existing node pair voltages in terms of these selected variables.

For simple networks involving a few elements, there is no difficulty in selecting the independent branch currents or the independent node-pair voltages. The set of linearly independent equations can be written by inspection. However for large scale networks particularly modern electronic circuits such as integrated circuits and microcircuits with a larger number of interconnected branches, it is almost impossible to write a set of linearly independent equations by inspection or by mere intuition. The problem becomes quite difficult and complex. A systematic and step by step method is therefore required to deal with such networks. Network topology (graph theory approach) is used for this purpose. By this method, a set of linearly independent loop or node equations can be written in a form that is suitable for a computer solution.

2.2 Terms and definitions

The description of networks in terms of their geometry is referred to as network topology. The adequacy of a set of equations for analyzing a network is more easily determined topologically than algebraically.

**Graph (or linear graph):** A network graph is a network in which all nodes and loops are retained but its branches are represented by lines. The voltage sources are replaced by short circuits and current sources are replaced by open circuits. (Sources without internal impedances or admittances can also be treated in the same way because they can be shifted to other branches by E-shift and/or I-shift operations.)

**Branch:** A line segment replacing one or more network elements that are connected in series or parallel.
**Node**: Interconnection of two or more branches. It is a terminal of a branch. Usually interconnections of three or more branches are nodes.

**Path**: A set of branches that may be traversed in an order without passing through the same node more than once.

**Loop**: Any closed contour selected in a graph.

**Mesh**: A loop which does not contain any other loop within it.

**Planar graph**: A graph which may be drawn on a plane surface in such a way that no branch passes over any other branch.

**Non-planar graph**: Any graph which is not planar.

**Oriented graph**: When a direction to each branch of a graph is assigned, the resulting graph is called an oriented graph or a directed graph.

**Connected graph**: A graph is connected if and only if there is a path between every pair of nodes.

**Sub graph**: Any subset of branches of the graph.

**Tree**: A connected sub-graph containing all nodes of a graph but no closed path. i.e. it is a set of branches of graph which contains no loop but connects every node to every other node not necessarily directly. A number of different trees can be drawn for a given graph.

**Link**: A branch of the graph which does not belong to the particular tree under consideration. The links form a sub-graph not necessarily connected and is called the co-tree.

**Tree compliment**: Totality of links i.e. Co-tree.

**Independent loop**: The addition of each link to a tree, one at a time, results one closed path called an independent loop. Such a loop contains only one link and other tree branches. Obviously, the number of such independent loops equals the number of links.

**Tie set**: A set of branches contained in a loop such that each loop contains one link and the remainder are tree branches.

**Tree branch voltages**: The branch voltages may be separated in to tree branch voltages and link voltages. The tree branches connect all the nodes. Therefore if the tree branch voltages are forced to be zero, then all the node potentials become coincident and hence all branch voltages are forced to be zero. As the act of setting only the tree branch voltages to zero forces all voltages in the network to be zero, it must be possible to express all the link voltages uniquely in terms of tree branch voltages. Thus tree branch form an independent set of equations.

**Cut set**: A set of elements of the graph that dissociates it into two main portions of a network such that replacing any one element will destroy this property. It is a set of branches that if removed divides a connected graph in to two connected sub-graphs. Each cut set contains one tree branch and the remaining being links.

Fig. 2.1 shows a typical network with its graph, oriented graph, a tree, co-tree and a non-planar graph.
Relation between nodes, links, and branches

Let \( B \) = Total number of branches in the graph or network
\( N \) = total nodes
\( L \) = link branches

Then \( N - 1 \) branches are required to construct a tree because the first branch chosen connects two nodes and each additional branch includes one more node.

Therefore number of independent node pair voltages = \( N - 1 \) = number of tree branches.
Then \( L = B - (N - 1) = B - N + 1 \)
Number of independent loops = \( B - N + 1 \)

2.3 Isomorphic graphs

Two graphs are said to be isomorphic if they have the same incidence matrix, though they look different. It means that they have the same numbers of nodes and the same numbers of branches. There is one to one correspondence between the nodes and one to one correspondence between the branches. Fig. 2.2 shows such graphs.

![Network Topology](image)

Figure 2.1

Figure 2.2


2.4 Matrix representation of a graph

For a given oriented graph, there are several representative matrices. They are extremely important in the analytical studies of a graph, particularly in the computer aided analysis and synthesis of large scale networks.

2.4.1 Incidence Matrix $A_n$

It is also known as augmented incidence matrix. The element node incidence matrix $A$ indicates in a connected graph, the incidence of elements to nodes. It is an $N \times B$ matrix with elements of $A_n = (a_{kj})$

\[
    a_{kj} = \begin{cases} 
        1, & \text{when the branch } b_j \text{ is incident to and oriented away from the } k^{th} \text{ node.} \\
        -1, & \text{when the branch } b_j \text{ is incident to and oriented towards the } k^{th} \text{ node.} \\
        0, & \text{when the branch } b_j \text{ is not incident to the } k^{th} \text{ node.}
    \end{cases}
\]

As each branch of the graph is incident to exactly two nodes,\[
    \sum_{k=0}^{n} a_{kj} = 0 \text{ for } j = 1, 2, 3, \ldots, B.
\]

That is, each column of $A_n$ has exactly two non zero elements, one being +1 and the other –1. Sum of elements of any column is zero. The columns of $A_n$ are linerally dependent. The rank of the matrix is less than $N$.

Significance of the incidence matrix lies in the fact that it translates all the geometrical features in the graph into an algebraic expression.

Using the incidence matrix, we can write $KCL$ as

$A_n \mathbf{i}_B = 0$, where $\mathbf{i}_B$ = branch current vector.

But these equations are not linearly independent. The rank of the matrix $A$ is $N - 1$. This property of $A_n$ is used to define another matrix called reduced incidence matrix or bus incidence matrix.

For the oriented graph shown in Fig. 2.3(a), the incidence matrix is as follows:

\[
    A_n = \begin{bmatrix}
        a & b & c & d \\
        1 & 2 & 3 & 4 & 5 \\
    \end{bmatrix}
\]

Note that sum of all elements in each column is zero.

Figure 2.3(a)
2.4.2 Reduced incidence matrix

Any node of a connected graph can be selected as a reference node. Then the voltages of the other nodes (referred to as buses) can be measured with respect to the assigned reference. The matrix obtained from \( A_n \) by deleting the row corresponding to the reference node is the element-bus incident matrix \( A \) and is called bus incidence matrix with dimension \((N - 1) \times B\). \( A \) is rectangular and therefore singular.

In \( A_n \), the sum of all elements in each column is zero. This leads to an important conclusion that if one row is not known in \( A \), it can be found so that sum of elements of each column must be zero.

From \( A \), we have \( A \cdot i_B = 0 \), which represents a set of linearly independent equations and there are \( N - 1 \) independent node equations.

For the graph shown in Fig 2.3(a), with \( d \) selected as the reference node, the reduced incidence matrix is

\[
A = \begin{bmatrix}
a & b & c \\
\begin{bmatrix}
-1 & 1 & -1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & -1 & -1 \\
\end{bmatrix}
\end{bmatrix}
\]

Note that the sum of elements of each column in \( A \) need not be zero.

Note that if branch current vector, \( j_B = \begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \end{bmatrix} \), then \( A \cdot i_B = 0 \) representing a set of independent node equations.

Another important property of \( A \) is that determinant \( A^T A \) gives the number of possible trees of the network. If \( A = [A_t : A_i] \) where \( A_t \) and \( A_i \) are sub-matrices of \( A \) such that \( A_t \) contains only twigs, then \( \det A_t \) is either +1 or -1.

To verify the property that \( \det A^T A \) gives the number of all possible trees, consider the reduced incidence matrix \( A \) of the example considered. That is,

\[
A = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
-1 & 1 & -1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & -1 & -1 \\
\end{bmatrix}
\]

Then, \( A \cdot i_B = 0 \) representing a set of independent node equations.

To verify the property, we can calculate \( \det A^T A \).

\[
\det A^T A = \begin{vmatrix}
-1 & 1 & -1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & -1 & -1 \\
\end{vmatrix}^T + \begin{vmatrix}
0 & 0 & 1 & 0 \\
-1 & -1 & 0 & 1 \\
\end{vmatrix}^T = 8
\]
Fig. 2.3(b) shows all possible trees corresponding to the matrix $A$.

To verify the property that the determinant of sub matrix $A_i$ of $A = A_t$; $A_i$ is +1 or −1.

For tree [2, 3, 4]

From

$$A = \begin{bmatrix} a & +1 & 1 & 0 & -1 & 0 \\ b & 0 & 0 & 1 & 1 & 0 \\ c & -1 & 0 & -1 & 0 & -1 \end{bmatrix} = A_i; A_i$$

$\det A_i = \begin{vmatrix} 1 & +1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & -1 \end{vmatrix} = 1$

For another tree [2, 4, 5]

$$A = \begin{bmatrix} a & 1 & 0 & 0 & -1 & -1 \\ b & 0 & 1 & 0 & 1 & 0 \\ c & -1 & -1 & -1 & 0 & 0 \end{bmatrix} = A_i; A_i$$

$\det A_i = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & -1 \end{vmatrix} = -1$

2.5 Loop equations and fundamental loop matrix (Tie-set Matrix)

From the knowledge of the basic loops (tie-sets), we can obtain loop matrix. In this matrix, the loop orientation is to be the same as the corresponding link direction. In order to construct this matrix, the following procedure is to be followed.
1. Draw the oriented graph of the network. Choose a tree.

2. Each link forms an independent loop. The direction of this loop is same as that of the corresponding link. Choose each link in turn.

3. Prepare the tie-set matrix with elements $b_{i,j}$,

where $b_{i,j} = 1$, when branch $b_j$ in loop $i$ and is directed in the same direction as the loop current.

$= -1$, when branch $b_j$ is in loop $i$ and is directed in the opposite direction as the loop current.

$= 0$, when branch $b_j$ is not in loop $i$.

Tie-set matrix is an $i \times b$ matrix.

Consider the example of Fig. 2.3(a).

![Diagram](image)

Figure 2.4

Selecting (2, 4, 5) as tree, the co-tree is (1, 3). Fig. 2.4 leads to the following tie-set.

$$
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
1 & 1 & 0 & -1 & 0 \\
0 & 1 & 1 & 0 & -1
\end{bmatrix}
$$

with

$$
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4 \\
v_5
\end{bmatrix} = V_B
$$

Then $MV_B$ gives the following independent loop equations:

$$
v_1 + v_2 - v_4 = 0
$$

$$
v_2 + v_3 - v_5 = 0
$$

Looking column wise, we can express branch currents in terms of loop currents. This is done by the following matrix equation.

$$
J_B = M^T I_L
$$
The above matrix equation gives $J_1 = i_1$, $J_2 = i_1 + i_2$, $J_3 = i_2$, $J_4 = -i_1 - i_2$

Note that $J$ stands for branch current while $i$ stands for loop current.

In this matrix,

(i) Each row corresponds to an independent loop. Therefore the columns of the resulting schedule automatically yield a set of equations relating each branch current to the loop currents.

(ii) As each column expresses a branch current in terms of loop currents, the rows of the matrix automatically yield the closed paths in which the associated loop currents circulate. Expressions for branch currents in terms of loop currents may be obtained in matrix form as $J_B = M^T I_L$.

where $M$ is the tie-set matrix of $L \times B$.

In the present example,

$$J_B = \begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \end{bmatrix}$$

and

$$I_L = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

### 2.6 Cut-set matrix and node pair potentials

A cut-set of a graph is a set of branches whose removal, cuts the connected graph into two parts such that the replacement of any one branch of the cut-set renders the two parts connected. For example, two separated graphs are obtained for the graph of Fig. 2.5(a) by selecting the cut-set consisting of branches \{1, 2, 5, 6\}. These separated graphs are as shown in Fig. 2.5(b).

Just as a systematic method exists for the selection of a set of independent loop current variables, a similar process exists for the selection of a set of independent node pair potential variables.
It is already known that the cut set is a minimal set of branches of the graph, removal of which divides the graph into two connected sub-graphs. Then it separates the nodes of the graph into two groups, each being one of the two sub-graphs. Each branch of the tie-set has one of its terminals incident at a node on one sub-graph. Selecting the orientation of the cut set same as that of the tree branch of the cut set, the cut set matrix is constructed row-wise taking one cut set at a time. Without link currents, the network is inactive. In the same way, without node pair voltages the network is active. This is because when one twig voltage is made active with all other twig voltages are zero, there is a set of branches which becomes active. This set is called cut-set. This set is obtained by cutting the graph by a line which cuts one twig and some links. The algebraic sum of these branch currents is zero. Making one twig voltage active in turn, we get entire set of node equations.

This matrix has current values,

\[ q_{i,j} = 1, \text{ if branch } J \text{ is in the cut-set with orientation same as that of tree branch.} \]
\[ = -1, \text{ if branch } J \text{ is in the cut-set with orientation opposite to that of tree branch.} \]
\[ = 0, \text{ if branch } J \text{ is not in the cut-set.} \]

and dimension is \((N - 1) \times B\).

Row-by-row reading, it gives the KCL at each node and therefore we have \(QJ_B = 0\).

The procedure to write cut-set matrix is as follows:

(i) Draw the oriented graph of a network and choose a tree.

(ii) Each tree branch forms an independent cut-set. The direction of this cut-set is same as that of the tree branch. Choose each tree branch in turn to obtain the cut set matrix. Isolate the tree element pairs and energize each bridging tree branch. Assuming the bridging tree branch potential equals the node pair potential, thus regarding it as an independent variable.

(iii) Use the columns of the cut-set matrix to yield a set of equations relating the branch potentials in terms of the node pair potentials. This may be obtained in matrix form as

\[ V_B = Q^T E_N \]

where \(v\) and \(e\) are used to indicate branch potential and node voltage respectively.

In the example shown in Fig 2.5 (c), \(3, 4, 5\) are tree branches. Links are shown in dotted lines. If two tree branch voltages in 3 and 4 are made zero, the nodes \(a\) and \(c\) are at the same potential. Similarly the nodes \(b\) and \(d\) are at the same potential. The graph is reduced to the form shown in Fig. 2.5(d) containing only the cut-set branches. Then, we have

\[ i_5 - i_1 - i_2 - i_6 = 0 \]

Similarly by making only \(e_4\) to exist (with \(e_5\) and \(e_3\) zero), the nodes \(a\), \(b\) and \(c\) are at the same potential, reducing the graph to the form shown in Fig. 2.5(e). Thus,

\[ i_4 + i_2 + i_6 = 0 \]
This corresponds to cut set 2 as shown in Fig. 2.5 (f).

For the remaining cut-set, $e_4$ and $e_5$ are made zero as in Fig. 2.5(g). $e_3$ is active and hence, the nodes $a$, $d$, and $c$ are at the same potential. Thus

$$i_1 + i_3 + i_2 = 0$$

The corresponding cut-set 3 is shown in Fig 2.5(h).

Therefore, the cut set schedule is

<table>
<thead>
<tr>
<th>Node pair</th>
<th>branches</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_1 = v_5$</td>
<td>$-1$ $-1$ $0$ $0$ $1$ $-1$</td>
</tr>
<tr>
<td>$e_2 = v_4$</td>
<td>$0$ $1$ $0$ $1$ $0$ $1$</td>
</tr>
<tr>
<td>$e_3 = v_3$</td>
<td>$1$ $1$ $1$ $0$ $0$ $0$</td>
</tr>
</tbody>
</table>

$$= Q$$
Note that $QJ_B$ gives the following equilibrium equations:

$$-J_1 - J_2 - J_5 + J_6 = 0$$
$$J_2 + J_4 + J_6 = 0$$
$$J_1 + J_2 + J_3 = 0$$

Looking column-wise, we can express branch voltages in terms of node pair voltages as

$$v_1 = -e_1 + e_3$$
$$v_2 = -e_1 + e_2 + e_3$$
$$v_3 = e_3$$
$$v_4 = e_2$$
$$v_5 = e_1$$
$$v_6 = -e_1 + e_2$$

Or

$$\begin{bmatrix}
-1 & 0 & 1 \\
-1 & 1 & 1 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
-1 & 1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2 \\
e_3 \\
\end{bmatrix} = V_B$$

That is $V_B = Q^T E_N$

### 2.7 Network Equilibrium Equations

**Loop equations:** A branch of a network can, in general, be represented as shown in Fig. 2.6,

![Network Diagram](image)

where $E_B$ is the voltage source of the branch.

$I_B$ is the current source of the branch

$Z_B$ is the impedance of the branch

$J_B$ is the current in the branch

The voltage-current relation is then given by

$$V_B = (J_B + I_B) Z_B - E_B$$

For a general network with many branches, the matrix equation is

$$V_B = Z_B (J_B + I_B) - E_B$$

(2.1)

where $V_B, J_B, I_B,$ and $E_B$ are $B \times 1$ vectors and $Z_B$ is the branch impedance matrix of $B \times B$. 
Each row of the tie-set matrix corresponds to a loop and involves all the branches of the loop. As per KVL, the sum of the corresponding branch voltages may be equated to zero. That is

\[ \mathbf{M} \mathbf{V}_B = 0 \]  

(2.2)

where \( \mathbf{M} \) is the tie-set matrix.

In the same matrix, each column represents a branch current in terms of loop currents. Transposed \( \mathbf{M} \) is used to give the relation between branch currents and loop currents.

\[ \mathbf{J}_B = \mathbf{M}^T \mathbf{I}_L \]  

(2.3)

This equation is called loop transformation equation. Substituting equation (2.1) in (2.2), we get

\[ \mathbf{MZ}_B \{ \mathbf{J}_B + \mathbf{I}_B \} - \mathbf{ME}_B = 0 \]  

(2.4)

Substituting equation (2.3) in (2.4), we get

\[ \mathbf{MZ}_B \mathbf{M}^T \mathbf{I}_L + \mathbf{MZ}_B \mathbf{I}_B - \mathbf{ME}_B = 0 \]

or

\[ \mathbf{MZ}_B \mathbf{M}^T \mathbf{I}_L = \mathbf{ME}_B - \mathbf{MZ}_B \mathbf{I}_B \]

With \( \mathbf{MZ}_B \mathbf{M}^T = \mathbf{Z}_L \), we have \( \mathbf{Z}_L \mathbf{I}_L = \mathbf{E}_L \)

If there are no current sources in the network, then

\[ \mathbf{Z}_L \mathbf{I}_L = \mathbf{E}_L \]  

(2.5)

where \( \mathbf{Z}_L \) is the loop impedance matrix and \( \mathbf{E}_L \) is the resultant loop voltage source vector.

**Node equations:** Next, each row of the cut-set matrix corresponds to a particular node pair voltage and indicates different branches connected to a particular node. KCL can be applied to the node and the algebraic sum of the branch currents at that node is zero.

\[ \mathbf{QJ}_B = 0 \]  

(2.6)

Each column of cut-set matrix relates a branch voltage to node pair voltages. Hence,

\[ \mathbf{V}_B = \mathbf{Q}^T \mathbf{E}_N \]  

(2.7)

This equation is known as node transformation equation. Current voltage relation for a branch is

\[ \mathbf{J}_B = \mathbf{Y}_B (\mathbf{V}_B + \mathbf{E}_B) - \mathbf{I}_B \]

For a network with many branches the above equation may be written in matrix form as

\[ \mathbf{J}_B = \mathbf{Y}_B \mathbf{V}_B + \mathbf{Y}_B \mathbf{E}_B - \mathbf{I}_B \]  

(2.8)

where \( \mathbf{Y}_B \) is branch admittance matrix of \( B \times B \).
The matrix nodal equations may be obtained from equations 2.6, 2.7, and 2.8. Substituting equation (2.8) in (2.6)

\[ Q Y_B V_B + Q Y_B E_B - Q I_B = 0 \]  

(2.9)

Substituting equation (2.7) in (2.9)

\[ Q Y_B Q^T E_N = Q (I_B - Y_B E_B) \]

In the absence of voltage sources, the equation becomes

\[ Y_N E_N = Q I_B = I_N \]

where \( Y_N \) is the node admittance matrix and \( I_N \) is the node current vector.

**Worked Examples**

**EXAMPLE 2.1**

Refer the circuit shown in Fig. 2.7(a). Draw the graph, one tree and its co-tree.

![Diagram](image)

**SOLUTION**

We find that there are four nodes \( N = 4 \) and seven branches \( B = 7 \). The graph is then drawn and appears as shown in Fig. 2.7 (b). It may be noted that node \( d \) represented in the graph (Fig. 2.7(b)) represents both the nodes \( d \) and \( e \) of Fig. 2.7(a). Fig. 2.7(c) shows one tree of graph shown in Fig. 2.7(b). The tree is made up of branches 2, 5 and 6. The co-tree for the tree of Fig. 2.7(c) is shown in Fig. 2.7(d). The co-tree has \( L = B - N + 1 = 7 - 4 + 1 = 4 \) links.
EXAMPLE 2.2

Refer the network shown in Fig. 2.8(a). Obtain the corresponding incidence matrix.

SOLUTION

The network shown in Fig. 2.8(a) has five nodes and eight branches. The corresponding graph appears as shown in Fig. 2.8(b).

The incidence matrix is formed by following the rule: The entry of the incidence matrix, $a_k = 1$, if the current of branch $k$ leaves the node $i$

$= -1$, if the current of branch enters node $i$

$= 0$, if the branch $k$ is not connected with node $i$.

Incidence matrix:

<table>
<thead>
<tr>
<th>Nodes</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>$e$</th>
<th>$f$</th>
<th>$g$</th>
<th>$h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>+1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td>+1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>-1</td>
<td>+1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>+1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>+1</td>
<td>+1</td>
<td>+1</td>
<td>+1</td>
</tr>
</tbody>
</table>
Some times, we represent an incidence matrix as follows:

\[
A_5 = \begin{bmatrix}
+1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\
-1 & +1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & -1 & +1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & +1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & +1 & +1 & +1 & +1 \\
\end{bmatrix}
\]

where subscript 5 indicates that there are five nodes in the graph. It may be noted that from a given incidence matrix, the corresponding graph can be drawn uniquely.

**EXAMPLE 2.3**

For the incidence matrix shown below, draw the graph.

\[
a = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
b & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\
c & 0 & 0 & 1 & 0 & 0 & -1 & 1 & -1 \\
d & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\
\end{bmatrix}
\]

**SOLUTION**

![Figure 2.9](image)

Observing the matrix, it can be seen that it is a reduced incidence matrix. Branches 1, 2, 3 and 4 are to be connected to the reference node. Branch 5 appears between the nodes \(a\) and \(b\), 6 between \(b\) and \(c\), 7 between \(c\) and \(d\) and 8 between \(a\) and \(c\). With this information, the oriented graph is drawn as shown in Fig. 2.9. Orientation is +1 for an arrow leaving a node \(d\) — 1 for an arrow entering a node.

**EXAMPLE 2.4**

Draw the graph of a network of whose the incidence matrix is as shown below.

\[
p = \begin{bmatrix}
0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
q & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\
r & -1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \\
s & 1 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 \\
\end{bmatrix}
\]
SOLUTION
Sum of the elements in columns 4, 9 are not zero. Therefore the given matrix is a reduced matrix. Taking 0 as reference node, the oriented graph is as shown in Fig. 2.10 after making the nodes in an order.

EXAMPLE 2.5
For the graph shown in Fig. 2.11(a), write the incidence matrix. Express branch voltage in terms of node voltages and then write a loop matrix and express branch currents in terms of loop currents.

SOLUTION
With the orientation shown in Fig. 2.11(a), the incidence matrix is prepared as shown below.

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 1 & -1 \\
-1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\
\end{bmatrix}
\]

Branch voltages in terms of node voltages are
\[
\begin{align*}
\v_1 &= e_a - e_d \\
\v_2 &= e_b - e_d \\
\end{align*}
\]

For loop (tie-set) matrix, \(L = B - N + 1 = 8 - 5 + 1 = 4\). With twigs (1, 2, 3, 4), we have chords (links) (5, 6, 7, 8) and the corresponding tree is as shown in Fig 2.11 (b). Introducing the chords one at a time, the tie-set matrix is prepared as shown below.
The branch currents in terms of loop currents are $J_1 = -i_1 - i_4$, $J_2 = i_1 - i_2$ etc.

**EXAMPLE 2.6**

For the network shown in Fig. 2.12(a), determine the number of all possible trees. For a tree consisting of (1, 2, 3) (i) draw tie set matrix (ii) draw cut-set matrix.

**SOLUTION**

If the intention is to draw a tree only for the purpose of tie-set and cut-set matrices, the ideal current source is open circuited and ideal voltage source is short circuited. The oriented graph is drawn for which d is the reference. Refer Fig. 2.12(b),

$$
A = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 0 & 0 & -1 & 1 & -1 \\
-1 & 1 & 0 & 1 & -1 & 0 \\
0 & -1 & 1 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\text{Det } AA^T = \begin{vmatrix}
1 & -1 & 0 \\
0 & 1 & 1 \\
-1 & 0 & 1 \\
0 & -1 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{vmatrix}
$$
Therefore, possible number of trees = 12.

Fig. 2.12(b) shows the corresponding graph, tree, co-tree, and loops 1, 2, 3.

(i) Tie-set matrix for twigs (1, 2, 3) is

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & -1 & 0 & -1 & 1 & -1 & 1 & 0 & 0 \\
-1 & 1 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 1 & -1 & 0 \\
0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\
\end{pmatrix}
\]

\[= 12\]

(ii) Cut-set matrix is

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
i_1 = J_4 & 1 & 1 & 1 & 0 & 0 \\
i_2 = J_5 & 1 & 0 & 0 & -1 & 0 \\
i_3 = J_6 & 1 & 1 & 1 & 0 & 1 \\
\end{pmatrix}
\]
**EXAMPLE 2.7**

For the network shown in Fig. 2.13(a), write a tie-set schedule and then find all the branch currents and voltages.

**SOLUTION**

Fig. 2.13(b) shows the graph for the network shown in Fig. 2.13(a). Also, a possible tree and co-tree are shown in Fig. 13(c). Co-tree is in dotted lines.
First, the tie-set schedule is formed and then the tie-set matrix is obtained.

**Tie-set schedule:**

<table>
<thead>
<tr>
<th>Loop currents</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>+1</td>
<td>0</td>
<td>0</td>
<td>+1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>( y )</td>
<td>0</td>
<td>+1</td>
<td>0</td>
<td>0</td>
<td>+1</td>
<td>-1</td>
</tr>
<tr>
<td>( z )</td>
<td>0</td>
<td>0</td>
<td>+1</td>
<td>-1</td>
<td>0</td>
<td>+1</td>
</tr>
</tbody>
</table>

**Tie-set matrix is**

\[
M = \begin{bmatrix}
1 & 0 & 0 & 1 & -1 & 0 \\
0 & 1 & 0 & 0 & 1 & -1 \\
0 & 0 & 1 & -1 & 0 & 1 \\
\end{bmatrix}
\]

The branch impedance matrix is

\[
Z_B = \begin{bmatrix}
5 & 0 & 0 & 0 & 0 & 0 \\
0 & 10 & 0 & 0 & 0 & 0 \\
0 & 0 & 5 & 0 & 0 & 0 \\
0 & 0 & 0 & 10 & 0 & 0 \\
0 & 0 & 0 & 0 & 5 & 0 \\
0 & 0 & 0 & 0 & 0 & 5 \\
\end{bmatrix}
\]

The loop impedance matrix is

\[
Z_L = MZ_B M^T
\]

\[
= \begin{bmatrix}
1 & 0 & 0 & 1 & -1 & 0 \\
0 & 1 & 0 & 0 & 1 & -1 \\
0 & 0 & 1 & -1 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
5 & 0 & 0 & 0 & 0 & 0 \\
0 & 10 & 0 & 0 & 0 & 0 \\
0 & 0 & 5 & 0 & 0 & 0 \\
0 & 0 & 0 & 10 & 0 & 0 \\
0 & 0 & 0 & 0 & 5 & 0 \\
0 & 0 & 0 & 0 & 0 & 5 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 & 0 & 1 & -1 & 0 \\
0 & 1 & 0 & 0 & 1 & -1 \\
0 & 0 & 1 & -1 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
5 & 0 & 0 & 0 & 0 & 0 \\
0 & 10 & 0 & 0 & 0 & 0 \\
0 & 0 & 5 & 0 & 0 & 0 \\
0 & 0 & 0 & 10 & 0 & 0 \\
0 & 0 & 0 & 0 & 5 & 0 \\
0 & 0 & 0 & 0 & 0 & 5 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 & 0 & 1 & -1 & 0 \\
0 & 1 & 0 & 0 & 1 & -1 \\
0 & 0 & 1 & -1 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
5 & 0 & 0 & 0 & 0 & 0 \\
0 & 10 & 0 & 0 & 0 & 0 \\
0 & 0 & 5 & 0 & 0 & 0 \\
0 & 0 & 0 & 10 & 0 & 0 \\
0 & 0 & 0 & 0 & 5 & 0 \\
0 & 0 & 0 & 0 & 0 & 5 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & -1 & 0 \\
0 & 1 & -1 \\
-1 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
10 & 0 & 0 \\
0 & 5 & 0 \\
0 & 0 & 5 \\
\end{bmatrix}
\begin{bmatrix}
0 & 0 & -1 \\
-1 & 1 & 0 \\
0 & -1 & 1 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
5 & 0 & 0 \\
0 & 10 & 0 \\
0 & 0 & 5 \\
\end{bmatrix}
+ \begin{bmatrix}
15 & -5 & -10 \\
-5 & 10 & -5 \\
-10 & -5 & 15 \\
\end{bmatrix}
= \begin{bmatrix}
20 & -5 & -10 \\
-5 & 20 & -5 \\
-10 & -5 & 20 \\
\end{bmatrix}
\]

\[
ME_B = \begin{bmatrix}
50 \\
0 \\
0 \\
\end{bmatrix}
\]
The loop equations are obtained using the equation,

\[
Z_L I_L = M E_B
\]

\[
\begin{bmatrix}
20 & -5 & -10 \\
-5 & 20 & -5 \\
-10 & -5 & 20 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix} =
\begin{bmatrix}
50 \\
0 \\
0 \\
\end{bmatrix}
\]

Solving by matrix method, we get

\[
x = 4.1666 \text{ A}, \quad y = 1.16666 \text{ A}, \quad z = 2.5 \text{ A}
\]

The branch currents are computed using the equations:

\[
I_B = M^T I_L
\]

\[
\begin{bmatrix}
I_1 \\
I_2 \\
I_3 \\
I_4 \\
I_5 \\
I_6 \\
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & -1 \\
-1 & 1 & 0 \\
0 & -1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix}
\]

Hence,

\[
I_1 = x = 4.1666 \text{ A}, \quad I_2 = y = 1.16666 \text{ A}, \quad I_3 = z = 2.5 \text{ A},
\]

\[
I_4 = x - z = 1.6666 \text{ A}, \quad I_5 = -x + y = -2.5 \text{ A}, \quad I_6 = -y + z = 0.8334 \text{ A}
\]

The branch voltages are computed using the equation:

\[
V_B = Z_B I_B - E_B
\]

\[
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
V_4 \\
V_5 \\
V_6 \\
\end{bmatrix} =
\begin{bmatrix}
5 & 0 & 0 & 0 & 0 & 0 \\
0 & 10 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 5 & 0 & 0 \\
0 & 0 & 0 & 0 & 10 & 0 \\
0 & 0 & 0 & 0 & 0 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2 \\
I_3 \\
I_4 \\
I_5 \\
I_6 \\
\end{bmatrix} =
\begin{bmatrix}
-50 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

Hence,

\[
V_1 = 5I_1 - 50 = 29.167 \text{ V}, \quad V_2 = 10I_2 = 16.666 \text{ V}, \quad V_3 = 5I_3 = 12.50 \text{ V},
\]

\[
V_4 = 10I_4 = 16.666 \text{ V}, \quad V_5 = 5I_5 = -12.50 \text{ V}, \quad V_6 = 5I_6 = 4.167 \text{ V}
\]
In the circuit, 4 Ω in series with current source is shorted (as it is trivial), the graph is as shown in Fig. 2.14(b) with 1 as tree branch and 2 as link.

Using the equation,

$$MZ_B M^T I_L = ME_B - MZ_B I_B$$

with $M = [1 \ 1]$, we have

$$MZ_B M^T = [1 \ 1] \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 8$$

$$ME_B - Z_B I_B = \begin{bmatrix} 1 & 1 \end{bmatrix} \left( \begin{bmatrix} 10 \\ 0 \end{bmatrix} - \begin{bmatrix} 4 & 4 \\ 0 & \frac{V_x}{4} \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ -V_x \end{bmatrix} = 10 - V_x$$

$$8I_1 = 10 - V_x \quad \text{but} \quad V_x = 4I_1$$

$$\Rightarrow \quad 8I_1 = 10 - 4I_1 \quad \Rightarrow \quad I_1 = \frac{5}{6}A$$

As $I_1 = \frac{V_x}{4} - I_2$, $I_2 = \frac{5}{6}A$

**Example 2.9**

For the given cut-set matrix, draw the oriented graph

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

**Solution**

Data from the matrix: $B = 7$, $N_t = 5$, $N = 4$, $L = 3$. Tree branch voltages are $e_1 = v_1$, $e_2 = v_2$, $e_3 = v_3$ and $e_4 = v_4$.

Therefore all these are connected to reference node. Individual cut-sets are

which indicates that 5 is common to $a$ and $b$, 7 is common to $b$ and $c$, 6 is common to $c$ and $d$. 
Therefore the graph is

![Network Topology Diagram](image)

**EXAMPLE 2.10**

Refer the network shown in Fig. 2.15(a). Solve for branch currents and branch voltages.

![Network Topology Diagram](image)

**SOLUTION**

The oriented graph for the network is shown in Fig. 2.15(b). A possible *tree* and *cotree* with fundamental cut-sets are shown in Fig. 2.15(c).

![Network Topology Diagram](image)
Cut-set schedule:

<table>
<thead>
<tr>
<th>Tree branch voltages</th>
<th>Branch Numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_1$</td>
<td>1  0  -1  0</td>
</tr>
<tr>
<td>$e_2$</td>
<td>0  1  -1  1</td>
</tr>
</tbody>
</table>

Cut-set matrix:

$$Q = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

Branch admittance matrix:

$$Y_B = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Cut-set admittance matrix:

$$Y_N = QY_BQ^T$$

$$= \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 \ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \ 0 & 1 \ -1 & -1 \ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$$

Equilibrium equations:

$$Y_N E_N = QI_B$$

$$\Rightarrow \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \end{bmatrix}$$

Solving using cramer’s rule, we get

$$e_1 = -\frac{70}{24} \text{ V}, \quad e_2 = \frac{20}{24} \text{ V}$$
Branch voltage are found using the matrix equation:

\[ V_B = Q^T E_N \]

\[
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
V_4
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
-1 & -1 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
-70 \\
24 \\
20 \\
24
\end{bmatrix}
\]

Hence,

\[ V_1 = \frac{-70}{24} \text{V} = -2.917 \text{V} \]
\[ V_2 = \frac{+20}{24} \text{V} = +0.833 \text{V} \]
\[ V_3 = \frac{70 - 20}{24} = +2.084 \text{V} \]
\[ V_4 = \frac{20}{24} = +0.833 \text{V} \]

Branch currents are found using the matrix equation (2.8):

\[ J_B = Y_B V_B - I_B \]

\[
\begin{bmatrix}
I_1 \\
I_2 \\
I_3 \\
I_4
\end{bmatrix}
= 
\begin{bmatrix}
2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 4
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
V_4
\end{bmatrix}
- 
\begin{bmatrix}
-10 \\
0 \\
0 \\
0
\end{bmatrix}
\]

\[
\begin{align*}
I_1 &= 2V_1 + 10 = 4.166 \text{ A} \\
I_2 &= V_2 = 0.833 \text{ A} \\
I_3 &= 2V_3 = 4.168 \text{ A} \\
I_4 &= 4V_4 = 3.332 \text{ A}
\end{align*}
\]

Verification:
Refer Fig. 2.15(a).

*KCL equations*

\[ I_1 = I_3 = 4.168 \text{ A} \]
\[ I_3 = I_2 + I_4 = 0.833 + 3.332 \]
\[ = 4.166 \text{ A} \]

and *KVL equations:*

\[ V_3 + V_2 + V_1 = 0 \]
\[ V_2 - V_4 = 0 \] are satisfied.
EXAMPLE 2.11

For the oriented graph shown, express loop currents in terms of branch currents for an independent set of columns as those pertinent to the links of a tree:

(i) Composed of 5, 6, 7, 8

(ii) Composed of 1, 2, 3, 6

Verify whether the two sets of relations for $i$’s in terms of $J$’s are equivalent. Construct a tie-set schedule with the currents in the links 4, 5, 7, 8 as loop currents and find the corresponding set of closed paths.

SOLUTION

For the first set

<table>
<thead>
<tr>
<th>Loop No.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>+1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>+1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>+1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>+1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>+1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>+1</td>
<td>-1</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>+1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>+1</td>
</tr>
</tbody>
</table>

Then for the second set, of the mesh currents indicated for the first set, we have

$J_4 = i_4$  \quad \Rightarrow \quad i_4 = J_4$

$J_5 = i_1 - i_4$  \quad \Rightarrow \quad i_1 = J_1 + J_5$

$J_7 = i_3 - i_2$  \quad \Rightarrow \quad i_3 = J_4 - J_8$

$J_8 = i_4 - i_3$  \quad \Rightarrow \quad i_2 = J_4 - J_7 - J_8$

By applying KCL for the oriented graph,

$i_1 = J_1 = J_4 + J_5; \quad i_2 = J_2 = J_3 - J_7 = J_4 - J_7 - J_8; \quad i_3 = J_3 = J_4 - J_8; \quad i_4 = J_4$

Thus the two sets of relations for $i$’s in terms of $J$’s are equivalent. The tie-set schedule with the currents in links 4, 5, 7, 8 as loop currents are shown below.
**EXAMPLE 2.12**

In the graph shown in Figure 2.16(a), the ideal voltage source $e = 1$ V. For the remaining branches each has a resistance of 1 $\Omega$ with $O$ as the reference. Obtain the node voltage $e_1$, $e_2$ and $e_3$ using network topology.

**SOLUTION**

With $e$ shift, graph is as shown in Figure 2.16(b). Branches are numbered with orientation.

With $T = (2, 5, 7)$ the cut set matrix is

$$ Q = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \end{bmatrix} $$
\[
\begin{align*}
Y_B Q^T &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
-1 & 0 & 0 \\
1 & 0 & 0 \\
1 & -1 & 0 \\
0 & -1 & 1 \\
0 & 0 & 1 \\
0 & 0 & -1 \\
\end{bmatrix} = \begin{bmatrix}
-1 & 0 & 0 \\
1 & 0 & 0 \\
1 & -1 & 0 \\
0 & -1 & 1 \\
0 & 0 & 1 \\
0 & 0 & -1 \\
\end{bmatrix},
\end{align*}
\]

\[
QY_B Q^T = \begin{bmatrix}
-1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & -1 \\
\end{bmatrix} \begin{bmatrix}
-1 & 0 & 0 \\
1 & 0 & 0 \\
1 & -1 & 0 \\
0 & -1 & 1 \\
0 & 0 & 1 \\
0 & 0 & -1 \\
\end{bmatrix} = \begin{bmatrix}
3 & -1 & 0 \\
-1 & 3 & -1 \\
0 & -1 & 3 \\
\end{bmatrix},
\]

\[
QY_B E_B = \begin{bmatrix}
-1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & -1 \\
\end{bmatrix} \begin{bmatrix}
1 \\
0 \\
0 \\
\end{bmatrix} = \begin{bmatrix}
-1 \\
0 \\
-1 \\
\end{bmatrix}
\]

According to the equation \(QY_B Q^T E_N = -QY_B E_B\), we have

\[
\begin{bmatrix}
3 & -1 & 0 \\
-1 & 3 & -1 \\
0 & -1 & 3 \\
\end{bmatrix} \begin{bmatrix}
e_1 \\
e_2 \\
e_3 \\
\end{bmatrix} = \begin{bmatrix} 1 \\
0 \\
1 \\
\end{bmatrix}
\]

Therefore,

\[
\begin{bmatrix}
e_1 \\
e_2 \\
e_3 \\
\end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\
-1 & 3 & -1 \\
0 & -1 & 3 \\
\end{bmatrix}^{-1} \begin{bmatrix} 1 \\
0 \\
1 \\
\end{bmatrix} = \begin{bmatrix} \frac{3}{7} \\
\frac{2}{7} \\
\frac{3}{7} \\
\end{bmatrix}
\]

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EXAMPLE 2.13

For the circuit shown in Fig. 2.17, for a tree consisting of \( ab, bc, cd \) form a tie-set schedule and obtain equilibrium loop equations. Choose branch numbers same as their resistance values. Solve for loop currents.

**Figure 2.17**

**SOLUTION**

The oriented graph is

Tree with nodes \( ab, bc, \) and \( cd \) and links are shown in dotted line. The tie set matrix is

\[
B \rightarrow \begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
0 & 1 & 1 & 0 & 1 & 0 \\
1 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 1 \\
\end{array}
\]

\[
M = \begin{bmatrix}
i_1 \\
i_2 \\
i_3 \\
\end{bmatrix}
\]
\[ M \mathbf{Z}_B \mathbf{M}^\top = \begin{bmatrix}
0 & 1 & 1 & 0 & 1 & 0 \\
1 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & -1 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & 0 & 5 \\
0 & 0 & 0 & 0 & 6
\end{bmatrix} \begin{bmatrix}
0 & 1 & 0 \\
1 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 4 & 0 \\
0 & 0 & 0 & 5 \\
0 & 0 & 0 & 0 & 6
\end{bmatrix} \begin{bmatrix}
0 & 1 & 0 \\
1 & -1 & 0 \\
1 & 0 & -1 \\
0 & 1 & -1 \\
1 & 0 & -1 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & 1 & 1 & 0 & 1 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 3
\end{bmatrix} \begin{bmatrix}
0 & 1 & 0 \\
1 & -1 & 0 \\
1 & 0 & 0
\end{bmatrix} 
+ \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 \\
-1 & -1 & 1
\end{bmatrix} \begin{bmatrix}
4 & 0 & 0 \\
0 & 5 & 0 \\
0 & 0 & 6
\end{bmatrix} \begin{bmatrix}
0 & 1 & -1 \\
1 & 0 & -1 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
5 & -2 & 0 \\
-2 & 3 & 0 \\
0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
5 & 0 & -5 \\
0 & 4 & -4 \\
-5 & -4 & 15
\end{bmatrix}
\]

\[
= \begin{bmatrix}
10 & -2 & -5 \\
-2 & 7 & -4 \\
-5 & -4 & 15
\end{bmatrix}
\]

\[ \mathbf{E}_B - \mathbf{Z}_B \mathbf{I}_B = \begin{bmatrix}
6 \\
4 \\
2 \\
-2 \\
0 \\
-6
\end{bmatrix} - \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 5 & 0 \\
0 & 0 & 0 & 0 & 0 & 6
\end{bmatrix} = \begin{bmatrix}
4 \\
4 \\
-3 \\
11 \\
0 \\
0
\end{bmatrix}
\]

\[ M(\mathbf{E}_B - \mathbf{Z}_B \mathbf{I}_B) = \begin{bmatrix}
0 & 1 & 1 & 0 & 1 & 0 \\
1 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & -1 & 1
\end{bmatrix} \begin{bmatrix}
2 \\
4 \\
11 \\
-2 \\
0 \\
-6
\end{bmatrix} = \begin{bmatrix}
15 \\
-4 \\
-4
\end{bmatrix}
\]

Then the loop equations,

\[ \begin{bmatrix}
10 & -2 & -5 \\
-2 & 7 & -4 \\
-5 & -4 & 15
\end{bmatrix} \begin{bmatrix}
i_1 \\
i_2 \\
i_3
\end{bmatrix} = \begin{bmatrix}
15 \\
-4 \\
-4
\end{bmatrix}
\]
The loop currents are

\[
\begin{bmatrix}
i_1 \\
i_2 \\
i_3
\end{bmatrix} = \begin{bmatrix} 10 & -2 & -5 \\ -2 & 7 & -4 \\ -5 & -4 & 15 \end{bmatrix}^{-1} \begin{bmatrix} 15 \\ -4 \\ -4 \end{bmatrix} = \frac{1}{575} \begin{bmatrix} 963 \\ 50 \\ 181 \end{bmatrix}
\]

**EXAMPLE 2.14**

For the network shown in Fig. 2.18(a), prepare a cut-set schedule and obtain equilibrium equations. Number the branches by their ohmic values.

**SOLUTION**

Numbering the branches same as those of ohmic values, the oriented graph is as shown in Fig. 2.18(b). Choosing 4, 5, 6 as tree branches, the tie set schedule is as shown below.
\[
Q = \begin{bmatrix}
e_a \\ e_b \\ e_c \end{bmatrix} = \begin{bmatrix}1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}
\]

\[
QY_BQ^T = \begin{bmatrix}
1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 1 & 0 & 0
\end{bmatrix} \begin{bmatrix}1^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 2^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 3^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 4^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 5^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 6^{-1}
\end{bmatrix} \begin{bmatrix}1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}1 & 1 & 0 \\ 0 & -1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix}1^{-1} & 0 & 0 \\ 0 & 2^{-1} & 0 \\ 0 & 0 & 3^{-1}
\end{bmatrix} \begin{bmatrix}1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}
\]

\[
+ \begin{bmatrix}0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix}4^{-1} & 0 & 0 \\ 0 & 5^{-1} & 0 \\ 0 & 0 & 6^{-1}
\end{bmatrix} \begin{bmatrix}0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}1 & 1 & 0 \\ 0 & -1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix}1^{-1} & 0 & 0 \\ 0 & 2^{-1} & 0 \\ 0 & 0 & 3^{-1}
\end{bmatrix} \begin{bmatrix}1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}1 & 1 & 0 \\ 0 & -1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix}1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}
\]

\[
I_B - Y_B E_B = \begin{bmatrix}-5 \\ 0 \\ -2 \\ -6 \end{bmatrix} - \begin{bmatrix}1^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 2^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 3^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 4^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 5^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 6^{-1}\end{bmatrix} \begin{bmatrix}0 \\ -6 \\ -21 \\ -140 \\ -16 \\ -2 \end{bmatrix} = \begin{bmatrix}-5 \\ 3 \\ 7 \\ 35 \\ -2 \\ -6\end{bmatrix}
\]

\[
Q(I_B - Y_B E_B) = \begin{bmatrix}1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix}-5 \\ 3 \\ 7 \\ 35 \\ -2 \\ -6\end{bmatrix} = \begin{bmatrix}-4 \\ -16 \\ 47\end{bmatrix}
\]
Equations in matrix form therefore is
\[
\begin{bmatrix}
1.7 & -0.5 & -1 \\
-0.5 & 1 & -0.33 \\
-1 & -0.33 & 1.58
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2 \\
e_3
\end{bmatrix}
= 
\begin{bmatrix}
-4 \\
-16 \\
47
\end{bmatrix}
\]

Without matrix method:
From the matrix \(Q\), looking row wise, we have by \(KCL\)
\[
J_1 + J_2 + J_5 = 0 \\
-J_2 - J_3 + J_6 = 0 \\
-J_1 + J_3 + J_4 = 0
\]
Looking column wise, we have
\[
v_1 = e_a - e_c; \quad v_2 = e_a - e_b; \quad v_3 = e_b - e_c; \quad v_4 = e_c; \quad v_5 = e_a; \quad v_6 = e_b
\]
Writing set (a) in terms of branch voltages and then (b) branch voltage in terms of node pair voltages,
For (i)
\[
v_1 + \frac{v_2}{2} + \frac{v_5}{4} = -4 \\
e_a - e_c + \frac{e_a - e_b}{2} + \frac{e_a}{5} = -4 \]
\[
1.7e_a - 0.5e_b - e_c = -4 \quad \text{(2.10)}
\]
For (ii)
\[
-\frac{v_2}{2} - \frac{v_3}{3} + \frac{v_6}{6} + 6 = 0 \\
\frac{e_b - e_a}{2} + \frac{e_b - e_c}{3} + \frac{e_b}{6} = -16 \]
\[
-0.5e_a + e_b - 0.33e_c = -16 \quad \text{(2.11)}
\]
For (iii)
\[
-5 - v_1 + \frac{v_3}{3} + \frac{v_4}{4} = 0 \\
-v_1 + \frac{v_3}{3} + \frac{v_4}{4} = 47 \\
-e_a + e_c + \frac{e_c - e_b}{3} + \frac{e_c}{4} = 47 \\
e_a - 0.33e_b = 47 - \frac{19}{12}e_c = 47 \quad \text{(2.12)}
\]
(2.10), (2.11), and (2.12) are the required equations.
EXAMPLE 2.15
Device a tree for the circuit shown in Fig. 2.19(a), for which all the currents pass through 1 Ω. For this tree write the tie-set matrix to obtain equilibrium equations.

![Figure 2.19(a)](image)

SOLUTION
Performing "I shift" the network is redrawn as shown in Fig. 2.19(b).

![Figure 2.19(b)](image)

The oriented graph is as shown in Fig. 2.19(c) for which the tree is shown in Fig. 2.19(d) to satisfy the condition.

![Figure 2.19(c)](image)  ![Figure 2.19(d)](image)

With link 2, row \(a\) of tie-set is
\[
1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0
\]

With link 3, row \(b\) of tie-set is
\[
0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0
\]

With link 7 row \(c\) of tie-set is
\[
0 \\ 0 \\ 0 \\ -1 \\ 0 \\ -1 \\ 1
\]
So that

\[
M = \begin{bmatrix}
1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & -1 & 1
\end{bmatrix}
\]

\[
MZ_B M^T = \begin{bmatrix}
1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & -1 & 1
\end{bmatrix} \begin{bmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 10 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 16 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.5 & 0 & 0 & 0 \\
0 & 0 & 0 & 6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.8
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 1 & -1 \\
1 & 1 & 0 \\
1 & 1 & -1 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & -1 & 1
\end{bmatrix} \begin{bmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 10 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 16 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.5 & 0 & 0 & 0 \\
0 & 0 & 0 & 6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.8
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 1 & -1 \\
1 & 1 & 0 \\
1 & 1 & -1 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & -1 & 1
\end{bmatrix} \begin{bmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 10 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 16 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.5 & 0 & 0 & 0 \\
0 & 0 & 0 & 6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.8
\end{bmatrix} \begin{bmatrix}
1 & 1 & 0 \\
1 & 1 & -1 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & -1 & 1
\end{bmatrix} \begin{bmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 10 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 16 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.5 & 0 & 0 & 0 \\
0 & 0 & 0 & 6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.8
\end{bmatrix} \begin{bmatrix}
30 \\
20 \\
-10 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

\[
MZ_B I_B = \begin{bmatrix}
1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & -1 & 1
\end{bmatrix} \begin{bmatrix}
0 & 0 & 16 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1.5 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
1 & 1 & 0 \\
1 & 1 & -1 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
260 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
0 \\
-160 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
260 \\
-160 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

Matrix equation is

\[
\begin{bmatrix}
20.5 & 7 & -2.5 & [i_1] \\
7 & 23 & -1 & [i_2] \\
-2.5 & -1 & 3.3 & [i_3]
\end{bmatrix} = \begin{bmatrix}
260 \\
-160 \\
0
\end{bmatrix}
\]
EXAMPLE 2.16
For the circuit shown in Fig. 2.20(a), construct a tree in which \( v_1 \) and \( v_2 \) are tree branch voltages, write a cut-set matrix and obtain equilibrium equations. Solve for \( v_1 \).

![Figure 2.20(a)](image)

![Figure 2.20(b)](image)

**SOLUTION**

Performing \( E \) shift and \( I \) shift, the circuit is redrawn as shown in Fig. 2.20(b).

With \( v_1 \) and \( v_2 \) as tree branch voltages the graph is as shown in Fig. 2.20(c), with tree branches with full lines and the links in dotted lines.

\[
Q = \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 1
\end{bmatrix}
\]

\[
QG_BQ^T = \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
0.1 & 0 & 0 & 0 \\
0 & 0.05 & 0 & 0 \\
0 & 0 & 0.02 & 0 \\
0 & 0 & 0 & 0.2
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 1 \\
1 & 1
\end{bmatrix}
= \begin{bmatrix}
0.1 \\
0.05
\end{bmatrix} + \begin{bmatrix}
0.2 & 0.2 \\
0.2 & 0.2
\end{bmatrix} = \begin{bmatrix}
0.3 & 0.2 \\
0.2 & 0.27
\end{bmatrix}
\]

\[
I_B - G_EB = \begin{bmatrix}
-2 & 0.1 & 0 & 0 & 0 \\
-2 & 0 & 0.05 & 0 & 0 \\
0 & 0 & 0 & 0.02 & 0 \\
0 & 0 & 0 & 0 & 0.2
\end{bmatrix}
= \begin{bmatrix}
0 & 0 \\
-20 & 0 \\
0 & 0 \\
-80 & 16
\end{bmatrix}
\]

\[
QI_B - G_EB = \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
-2 \\
-1 \\
0 \\
16
\end{bmatrix}
= \begin{bmatrix}
14 \\
15
\end{bmatrix}
\]
Therefore the equilibrium equation is:

\[
\begin{bmatrix}
0.3 & 0.2 \\
0.2 & 0.27
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2
\end{bmatrix} =
\begin{bmatrix}
14 \\
15
\end{bmatrix}
\]

Therefore,

\[
\begin{bmatrix}
e_1 \\
e_2
\end{bmatrix} =
\begin{bmatrix}
0.3 & 0.2 \\
0.2 & 0.27
\end{bmatrix}^{-1}
\begin{bmatrix}
14 \\
15
\end{bmatrix} =
\begin{bmatrix}
19.024 \\
41.463
\end{bmatrix}
\]

**EXAMPLE 2.17**

(a) Construct a tree for the circuit shown in Fig. 2.21(a) so that \(i_1\) is a link current. Assign a complete set of link currents and find \(i_1(t)\).

(b) Construct another tree in which \(v_1\) is a tree branch voltage. Assign a complete set of tree branch voltages and find \(v_1(t)v\).

Take \(i(t) = 25\sin10^3t\) Amp, \(v(t) = 15\cos10^3(t)\)

**SOLUTION**

(a) The circuit after \(I\) shift is as shown in Fig. 2.21(b). The oriented graph is as shown in Fig. 2.21(c). With branches as numbered, and 1 as a link the tree is as shown in Fig. 2.21(d). Links are shown in dotted line (links are 1 and 4).

The tie-set matrix is:

\[
M = \begin{bmatrix}
1 & -1 & 1 & 0 \\
0 & -1 & 1 & 1
\end{bmatrix}
\]

The branch impedence matrix is:

\[
Z_B = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -j1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & j^2
\end{bmatrix}
\]

\[
E_B - Z_B I_B = \begin{bmatrix}
0 \\
0 \\
j15
\end{bmatrix} - \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -j1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
j15
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
25 \\
j15
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
-25 \\
j15
\end{bmatrix}
\]
\[
M (E_B - Z_B I_B) = \begin{bmatrix}
1 & -1 & 1 & 0 \\
0 & -1 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
-25 \\
j15
\end{bmatrix}
= \begin{bmatrix}
25 \\
-25 + j15
\end{bmatrix}
\]

\[
MZ_B M_T = \begin{bmatrix}
1 & -1 & 0 \\
0 & -1 & 1 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -j1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
-1 & -1 \\
1 & 1 \\
0 & 0 & j2 & 0
\end{bmatrix}
= \begin{bmatrix}
2 - j1 & 1 - j1 \\
1 - j1 & 1 + j1
\end{bmatrix}
\]

Equilibrium equations are

\[
\begin{bmatrix}
2 - j1 & 1 - j1 \\
1 - j1 & 1 + j1
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix}
= \begin{bmatrix}
25 \\
-25 + j15
\end{bmatrix}
\]

Therefore

\[
I_1 = \frac{25}{1 - j1}
\]

\[
= 15.72 \sin (10^3 t - 148^\circ) \text{ Amp}
\]

(b) With 1 as tree branch, the oriented tree is as shown in Fig. 2.21(c). Links are shown with dotted lines, with \(e_1\) and \(e_2\) as node pair voltages the cut-sets are shown.

The cut-set matrix is

\[
Y = \begin{bmatrix}
e_a & 1 & 0 & -1 & 1 \\
e_b & 0 & 1 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
j1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -j0.5
\end{bmatrix}
\]
\[ \mathbf{I}_B - \mathbf{Y}_B \mathbf{E}_B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & j1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -j0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ j15 \\ -j0.5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 25 \\ -7.5 \end{bmatrix} \]

\[ \mathbf{Q} (\mathbf{I}_B - \mathbf{Y}_B \mathbf{E}_B) = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 25 \\ -7.5 \end{bmatrix} = \begin{bmatrix} -32.5 \\ 25 \end{bmatrix} \]

\[ \mathbf{QY}_B \mathbf{Q}^T = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & j1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -j0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 + j1 \end{bmatrix} = \begin{bmatrix} 2 - j0.5 & -1 \\ -1 & 1 + j1 \end{bmatrix} \]

Equilibrium equations are

\[ \begin{bmatrix} 2 - j0.5 & -1 \\ -1 & 1 + j1 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} -32.5 \\ 25 \end{bmatrix} \]

Therefore

\[ E_1 = \frac{-32.5}{25} \begin{bmatrix} 2 - j0.5 \\ 1 + j1 \end{bmatrix} \]

\[ E_1 = 15.72/\angle148^\circ \text{ volts} \]

Therefore

\[ E_1 = 15.72 \sin (10^3t - 148^\circ) \text{ volts} \]

2.8 Dual Networks

Two electrical circuits are duals if the mesh equations that characterize one of them have the same mathematical form as the nodal equations that characterize the other.

Let us consider the series \( R_a - L_a - C_a \) network excited by a voltage source \( v_a \) as shown in Fig. 2.22(a), and the parallel \( G_b - C_b - L_b \) network fed by a current source \( i_b \) as shown in Fig. 2.22(b).

---

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The equations governing the circuit behaviour are:

(i)
\[
L_a \frac{di_a}{dt} + R_a i_a + \frac{1}{C_a} \int i_a dt = v_a
\]

\[
\Rightarrow L_a \frac{di_a}{dt} + R_a i_a + \frac{1}{C_a} = v_a
\]

\[
\Rightarrow L_a \frac{di_a}{dt} + R_a i_a + \frac{1}{C_a} \int i_a dt = v_a
\]

(ii)
\[
C_b \frac{dv_b}{dt} + G_b v_b + \frac{1}{L_b} \int v_b dt = i_b
\]

\[
\Rightarrow C_b \frac{dv_b}{dt} + G_b v_b + \frac{1}{L_b} \int v_b dt = i_b
\]

\[
\Rightarrow C_b \frac{dv_b}{dt} + G_b v_b + \frac{1}{L_b} \int v_b dt = i_b
\]

Comparing equations (2.13) and (2.14), we get the similarity between the networks of Fig. 2.22(a) and Fig 2.22(b). The solution of equation (2.13) will be identical to the solution of equation (2.14) when the following exchanges are made:

\[
L_a \rightarrow C_b, \quad R_a \rightarrow G_b, \quad C_a \rightarrow L_b
\]

\[
v_a \rightarrow i_b, \quad i_a \rightarrow v_b
\]

Hence, the series network in Fig. 2.22(a) and parallel network in Fig. 2.22(b) are duals of each other. The advantage of duality is that there is no need to analyze both types of circuits, since the solution of one automatically gives the solution of the other with a suitable change of symbols for the physical quantities. Table 2.1 gives the corresponding quantities for dual electrical networks.

<table>
<thead>
<tr>
<th>Loop basis</th>
<th>Node basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. A loop made up of several branches</td>
<td>1. A node joining the same number of branches.</td>
</tr>
<tr>
<td>2. Voltage sources</td>
<td>2. Current sources</td>
</tr>
<tr>
<td>3. Loop currents</td>
<td>3. Node voltages</td>
</tr>
<tr>
<td>4. Inductances</td>
<td>4. Capacitances</td>
</tr>
<tr>
<td>5. Resistances</td>
<td>5. Conductance</td>
</tr>
<tr>
<td>6. Capacitances</td>
<td>6. Inductances</td>
</tr>
</tbody>
</table>

Only planar networks have duals. The duals of planar networks could be obtained by a graphical technique known as the dot method. The dot method has the following procedure:

1. Put a dot in each independent loop of the network. These dots correspond to independent nodes in the dual network.

---

1 Planar networks are those that can be laid on a plane without branches crossing one another.
2. Put a dot outside the network. This dot corresponds to the reference node in the dual network.

3. Connect all internal dots in the neighbouring loops by dashed lines cutting the common branches. These branches that are cut by dashed lines will form the branches connecting the corresponding independent nodes in the dual network. As an example, if a common branch contains $R$ and $L$ in series, then the parallel combination of $G$ and $C$ should be put between the corresponding independent nodes in the dual network.

4. Join all internal dots to the external dot by dashed lines cutting all external branches. Duals of these branches cut by dashed lines will form the branches connecting the independent nodes and the reference node.

5. Convention for sources in the dual network:

   (i) a clockwise current source in a loop corresponds to a voltage source with a positive polarity at the dual independent node.

   (ii) a voltage rise in the direction of a clockwise loop current corresponds to a current flowing toward the dual independent node.

**EXAMPLE 2.18**

Draw the dual of the circuit shown in Fig. 2.23(a). Write the mesh equations for the given network and node equations for its dual. Verify whether they are dual equations.

![Figure 2.23(a)](image-url)
**SOLUTION**

For the given network, the mesh equations are

\[ R_1 i_1 + L_1 D (i_1 - i_2) + \frac{1}{C} \int (i_1 - i_3) \, dt = v_y \]

\[ i_2 = -i_0 \]

\[ R_2 i_3 + L_2 D i_3 + R_3 (i_3 - i_2) + \frac{1}{C} \int (i_3 - i_2) \, dt = 0 \]

The dual network, as per the procedure described in the theory is prepared as shown in Fig. 2.23(b) and is drawn as shown in Fig. 2.23(c). The node equations for this network are

\[ G_1 V_1 + C_1 D (v_1 - v_2) + \frac{1}{L} \int (v_1 - v_3) \, dt = i_y \]

\[ G_2 v_3 + C_2 D v_3 + G_3 (v_3 - v_2) + \frac{1}{L} \int (v_3 - v_2) \, dt = 0 \]
Example 2.19

For the bridge network shown in Fig. 2.24(a), draw its dual. Write the integro-differential form of the mesh equations for the given network and node equations for its dual. The values for resistors one in ohms, capacitors are in farads and inductors are in Henrys.

\[ 10i_1 + D(i_1 - i_2) + \frac{1}{4} \int (i_1 - i_3) \, dt = 10 \sin 50t \]
\[ D(i_2 - i_1) + 2Di_2 + \frac{1}{5} \int (i_2 - i_3) \, dt = 0 \]
\[ 3i_3 + \frac{4}{5} \int (i_3 - i_1) \, dt + \frac{1}{5} \int (i_3 - i_2) \, dt = 0 \]
The node equations for the dual network are

\[ 10v_1 + D(v_1 - v_2) + \frac{1}{4} \int (v_1 - v_3) \, dt = 10 \sin 50t \]

\[ D(v_2 - v_1) + 2Dv_2 + \frac{1}{5} \int (v_2 - v_3) \, dt = 0 \]

\[ 3v_3 + \frac{1}{4} \int (v_3 - v_1) \, dt + \frac{1}{5} \int (v_3 - v_2) \, dt = 0 \]

**EXAMPLE 2.20**

A network with a controlled source is shown in Fig. 2.25(a). Draw the dual for the given network and write equations for both the networks.

**SOLUTION**

The dual for the given network is shown in Fig. 2.25(c) using the procedure given in Fig. 2.25(b).
Mesh equations for the given network are

\[ i_x = i_1 - i_4 \]

\[ 5i_x + 10 \int (i_1 - i_2) \, dt = 2e^{-10t} \]

\[ i_2 - i_3 = -0.2i_x \]

\[ i_3 = -0.1e^{-10t} \]

\[ -5i_x + (i_4 - i_3) 20 + 10 \times 10^{-3} Di_4 = 0 \]

The node equations for the dual network are

\[ v_x = v_1 - v_4 \]

\[ 5v_x + 10 \int (v_1 - v_2) \, dt = 2e^{-10t} \]

\[ v_2 - v_3 = -0.2v_x \]

\[ v_3 = -0.1e^{-10t} \]

\[ -5v_x + (v_4 - v_3) 20 + 10 \times 10^{-3} Dv_4 = 0 \]

**Exercise Problems**

**E.P 2.1**

Consider the bridge circuit of Fig. E.P. 2.1. Using node \( d \) as the datum, determine the graph, select a tree, find the cut set equations, and determine \( v_a \).

**Ans:** 7.58 V
E.P. 2.2
Identify the 16 trees in the graph of Fig. E.P. 2.2.

Figure E.P. 2.2

E.P. 2.3
Refer the network shown in Fig. E.P. 2.3. The ohmic values also represent the branch numbers. Form a tree with tree branches 4, 5, 6 and find the various branch currents using the concept of tie-set and cut-set matrices.

Ans: \( I_1 = -24.114 \text{ A} \quad I_2 = 0.297 \text{ A} \quad I_3 = -4.5 \text{ A} \\
I_4 = -28.6112 \text{ A} \quad I_5 = 4.206 \text{ A} \quad I_6 = 24.407 \text{ A} \)
Refer the network shown in Fig.E.P. 2.4. Find the current supplied by the 10 V battery and the power dissipated in the 10-ohm resistor (connected across \(a - b\)) by using tie-set and cut-set matrices.

Ans: 1.047 A, 2.209 W

Refer the network shown in Fig. E.P. 2.5. Find the power dissipated by the 2 \(\Omega\) resistor by constructing tie-set and cut-set matrices.

Ans: 6.4 KW