**Lecture 1-5:**

**Two Dimensional State of Stress and Strain:** Principal stress examples

**Stresses on oblique plane:** Till now we have dealt with either pure normal direct stress or pure shear stress. In many instances, however both direct and shear stresses acts and the resultant stress across any section will be neither normal nor tangential to the plane. A plane state of stress is a 2 dimensional state of stress in a sense that the stress components in one direction are all zero i.e

\[ \sigma_z = \tau_{yz} = \tau_{zx} = 0 \]

Examples of plane state of stress include plates and shells. Consider the general case of a bar under direct load \( F \) giving rise to a stress \( \sigma_y \) vertically

The stress acting at a point is represented by the stresses acting on the faces of the element enclosing the point. The stresses change with the inclination of the planes passing through that point i.e. the stress on the faces of the element vary as the angular position of the element changes. Let the block be of unit depth now considering the equilibrium of forces on the triangle portion ABC. Resolving forces perpendicular to BC, gives

\[ \sigma_\theta \cdot BC \cdot 1 = \sigma_y \sin \theta \cdot AB \cdot 1 \]

but \( AB/BC = \sin \theta \) or \( AB = BC \sin \theta \)

Substituting this value in the above equation, we get
\[ \sigma_{\theta}.BC.1 = \sigma_y \sin \theta \cdot BC \sin \theta \cdot 1 \text{ or } \sigma_{\theta} = \sigma_y \sin^2 \theta \]  

(1)

**Now resolving the forces parallel to BC**

\[ \tau_{\theta}.BC.1 = \sigma_y \cos \theta \cdot AB \sin \theta \cdot 1 \]

again AB = BC \cos \theta

\[ \sigma_{\theta}.BC.1 = \sigma_y \cos \theta \cdot BC \sin \theta \cdot 1 \text{ or } \sigma_{\theta} = \sigma_y \sin \theta \cos \theta \]

\[ \tau_{\theta} = \frac{1}{2} \sigma_y \sin 2\theta \]  

(2)

If \( \theta = 90^0 \) the BC will be parallel to AB and \( \tau_{\theta} = 0 \), i.e. there will be only direct stress or normal stress.

By examining the equations (1) and (2), the following conclusions may be drawn

(i) The value of direct stress \( \sigma_{\theta} \) is maximum and is equal to \( \sigma_y \) when \( \theta = 90^0 \).

(ii) The shear stress \( \tau_{\theta} \) has a maximum value of 0.5 \( \sigma_y \) when \( \theta = 45^0 \)

**Material subjected to pure shear:**

Consider the element shown to which shear stresses have been applied to the sides AB and DC

Complementary shear stresses of equal value but of opposite effect are then set up on the sides AD and BC in order to prevent the rotation of the element. Since the applied and complementary shear stresses are of equal value on the x and y planes. Therefore, they are both represented by the symbol \( \tau_{xy} \).

Now consider the equilibrium of portion of PBC
Assuming unit depth and resolving normal to PC or in the direction of $\sigma_\theta$

$$\sigma_\theta \cdot PC \cdot 1 = \tau_{xy} \cdot PB \cdot \cos \theta \cdot 1 + \tau_{xy} \cdot BC \cdot \sin \theta \cdot 1$$

$$= \tau_{xy} \cdot PB \cdot \cos \theta + \tau_{xy} \cdot BC \cdot \sin \theta$$

Now writing PB and BC in terms of PC so that it cancels out from the two sides

$$PB/PC = \sin \theta \quad BC/PC = \cos \theta$$

$$\sigma_\theta \cdot PC \cdot 1 = \tau_{xy} \cdot \cos \theta \cdot \sin \theta \cdot PC + \tau_{xy} \cdot \cos \theta \cdot \sin \theta \cdot PC$$

$$\sigma_\theta = 2 \tau_{xy} \sin \theta \cos \theta$$

Or, $$\sigma_\theta = 2 \tau_{xy} \sin 2\theta \quad (1)$$

Now resolving forces parallel to PC or in the direction of $\sigma_\theta$ then $\tau_{xy} \cdot PC \cdot 1$

$$= \tau_{xy} \cdot PB \cdot \sin \theta - \tau_{xy} \cdot BC \cdot \cos \theta$$

-ve sign has been put because this component is in the same direction as that of $\tau_\theta$.

Again converting the various quantities in terms of PC we have

$$\tau_{xy} \cdot PC \cdot 1 = \tau_{xy} \cdot PB \cdot \sin^2 \theta - \tau_{xy} \cdot PC \cdot \cos^2 \theta$$

$$= -\tau_{xy} \left[ \cos^2 \theta - \sin^2 \theta \right]$$

$$= -\tau_{xy} \cos 2\theta \quad (2)$$

The negative sign means that the sense of $\tau_\theta$ is opposite to that of assumed one. Let us examine the equations (1) and (2) respectively

From equation (1) i.e,

$$\sigma_\theta = \tau_{xy} \sin 2\theta$$

The equation (1) represents that the maximum value of $\sigma_\theta$ is $\tau_{xy}$ when $\theta = 45^0$. Let us take into consideration the equation (2) which states that 41

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\[ \sigma_\theta = - \tau_{xy} \cos2 \theta \]

It indicates that the maximum value of \( \sigma_\theta \) is \( \tau_{xy} \) when \( \theta = 0^0 \) or \( 90^0 \). It has a value zero when \( \theta = 45^0 \).

From equation (1) it may be noticed that the normal component has maximum and minimum values of \( + \tau_{xy} \) (tension) and \( -\tau_{xy} \) (compression) on plane at \( \pm 45^0 \) to the applied shear and on these planes the tangential component is zero.

Hence the system of pure shear stresses produces and equivalent direct stress system, one set compressive and one tensile each located at \( 45^0 \) to the original shear directions as depicted in the figure below:

**Material subjected to two mutually perpendicular direct stresses:**

Now consider a rectangular element of unit depth, subjected to a system of two direct stresses both tensile, \( x \) and \( y \) acting right angles to each other.
for equilibrium of the portion ABC, resolving perpendicular to AC

\[ \sigma_\theta \cdot AC.1 = \sigma_y \sin \theta \cdot AB.1 + \sigma_x \cos \theta \cdot BC.1 \]

converting AB and BC in terms of AC so that AC cancels out from the sides

\[ \sigma_\theta = \sigma_y \sin^2 \theta + \sigma_x \cos^2 \theta \]

Further, recalling that \( \cos^2 \theta - \sin^2 \theta = \cos 2\theta \) or \( (1 - \cos 2\theta)/2 = \sin^2 \theta \)

Similarly \( (1 + \cos 2\theta)/2 = \cos^2 q \)

Hence by these transformations the expression for reduces to

\[ = \frac{1}{2} \ y (1 + \cos 2\theta) + \frac{1}{2} \ x (1 + \cos 2\theta) \]

On rearranging the various terms we get

\[ c_\theta = \left( \frac{\sigma_x + \sigma_y}{2} \right) + \left( \frac{\sigma_x - \sigma_y}{2} \right) \cos 2\theta \]

(3)

Now resolving parallel to AC

\[ s_q \cdot AC.1 = xy \cdot \cos AB.1 + xy \cdot BC.1 \]

The –ve sign appears because this component is in the same direction as that of AC.

Again converting the various quantities in terms of AC so that the AC cancels out from the two sides.

\[ \tau_\theta \cdot AC.1 = [\tau_x \cos \theta \sin \theta - c_y \sin \delta \cos \theta] \cdot AC \]

\[ \tau_\theta = (\sigma_x - \sigma_y) \sin \theta \cos \theta \]

\[ = \frac{(\sigma_x - \sigma_y)}{2} \sin 2\theta \]

(4)

Conclusions :

The following conclusions may be drawn from equation (3) and (4)

(i) The maximum direct stress would be equal to \( x \) or \( y \) which ever is the greater, when \( \theta = 0^0 \) or \( 90^0 \)

(ii) The maximum shear stress in the plane of the applied stresses occurs when \( \theta = 45^0 \)

\[ \tau_{\text{max}} = \frac{(\sigma_x - \sigma_y)}{2} \]
Material subjected to combined direct and shear stresses:

Now consider a complex stress system shown below, acting on an element of material.

The stresses $x$ and $y$ may be compressive or tensile and may be the result of direct forces or as a result of bending. The shear stresses may be as shown or completely reversed and occur as a result of either shear force or torsion as shown in the figure below:

As per the double subscript notation the shear stress on the face BC should be notified as $y_x$, however, we have already seen that for a pair of shear stresses there is a set of complementary shear stresses generated such that $y_x = x_y$.

By looking at this state of stress, it may be observed that this state of stress is combination of two different cases:

(i) Material subjected to pure state of stress shear. In this case the various formulas deserved are as follows

$$\sigma_\theta = y_x \sin^2 \theta$$
$$\sigma_\theta = y_x \cos^2 \theta$$

(ii) Material subjected to two mutually perpendicular direct stresses. In this case the various formula's derived are as follows.

$$\sigma_\delta = \frac{(\sigma_x - \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2} \cos 2\theta$$
$$\tau_\epsilon = \frac{(\sigma_x - \sigma_y)}{2} \sin 2\theta$$

To get the required equations for the case under consideration, let us add the respective equations for the above two cases such that
These are the equilibrium equations for stresses at a point. They do not depend on material proportions and are equally valid for elastic and inelastic behaviour.

This eqn gives two values of 2 that differ by $180^\circ$. Hence the planes on which maximum and minimum normal stresses occur are $90^\circ$ apart.

For $\sigma_\theta$ to be a maximum or minimum $\frac{d\sigma_\theta}{d\theta} = 0$

Now

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\frac{d\sigma_\theta}{d\theta} = -\frac{1}{2} (\sigma_x - \sigma_y) \sin 2\theta + \tau_{xy} \cos 2\theta$$

$$= 0$$

i.e. $-(\sigma_x - \sigma_y) \sin 2\theta + \tau_{xy} \cos 2\theta = 0$

$$\tau_{xy} \cos 2\theta = (\sigma_x - \sigma_y) \sin 2\theta$$

Thus,

$$\tan 2\theta = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$$

From the triangle it may be determined

$$\cos 2\theta = \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

$$\sin 2\theta = \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

Substituting the values of $\cos 2\theta$ and $\sin 2\theta$ in equation (5) we get
This shows that the values of shear stress is zero on the principal planes. Hence the maximum and minimum values of normal stresses occur on planes of zero shearing stress. The maximum and minimum normal stresses are called the principal stresses, and the planes on which they act are called principal planes.

The principal stresses are determined by solving the equation:

\[ \sigma = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y) \cos 2\theta + \tau_{xy} \sin 2\theta \]

or

\[ \sigma = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y) \cos 2\theta + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \]

Hence we get the two values of \( \sigma \), which are designated \( \sigma_1 \) as \( \sigma_2 \) and respectively, therefore

\[ \sigma_1 = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y) \cos 2\theta \]

\[ \sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}(\sigma_x - \sigma_y) \cos 2\theta \]

The \( \sigma_1 \) and \( \sigma_2 \) are termed as the principle stresses of the system.

Substituting the values of \( \cos 2\theta \) and \( \sin 2\theta \) in equation (6) we see that

\[ \tau_{\theta} = \frac{1}{2}(\sigma_x - \sigma_y) \sin 2\theta - \tau_{xy} \cos 2\theta \]

\[ \tau_{\theta} = \frac{1}{2}(\sigma_x - \sigma_y) \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} - \frac{\tau_{xy} (\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} \]

\[ \tau_{\theta} = 0 \]

This shows that the values of shear stress is zero on the principal planes.

Hence the maximum and minimum values of normal stresses occur on planes of zero shearing stress. The maximum and minimum normal stresses are called the principal stresses, and the planes on which they act are called principal plane the solution of equation

\[ \tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \]
will yield two values of 2 separated by 180° i.e. two values of separated by 90°. Thus the two principal stresses occur on mutually perpendicular planes termed principal planes.

Therefore the two – dimensional complex stress system can now be reduced to the equivalent system of principal stresses.

Let us recall that for the case of a material subjected to direct stresses the value of maximum shear stresses

\[ \tau_{\text{max}} = \frac{1}{2} (\sigma_x - \sigma_y) \text{ at } \theta = 45^\circ \]

Thus, for a 2-dimensional state of stress, subjected to principal stresses

\[ \tau_{\text{max}} = \frac{1}{2} (\sigma_1 - \sigma_2) \text{. on substituting the values of } \sigma_1 \text{ and } \sigma_2 \text{, we get} \]

\[ \tau_{\text{max}} = \frac{1}{2} \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \]

Alternatively, this expression can also be obtained by differentiating the expression for \( \tau_\theta \) with respect to \( \theta \) i.e.

\[ \tau_\theta = \frac{1}{2} (\sigma_x - \sigma_y) \sin 2\theta - \tau_{xy} \cos 2\theta \]

\[ \frac{d\tau_\theta}{d\theta} = \frac{1}{2} (\sigma_x - \sigma_y) \cos 2\theta + \tau_{xy} \sin 2\theta \]

Or \( (\sigma_x - \sigma_y) \cos 2\theta + 2\tau_{xy} \sin 2\theta = 0 \)

\[ \tan 2\theta_s = \frac{\sigma_y - \sigma_x}{2\tau_{xy}} \]

\[ \tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \]

Recalling that

\[ \tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \]

Thus,

\[ \tan 2\theta_p \cdot \tan 2\theta_s = 1 \]
Therefore, it can be concluded that the equation (2) is a negative reciprocal of equation (1) hence the roots for the double angle of equation (2) are 90° away from the corresponding angle of equation (1).

This means that the angles that locate the plane of maximum or minimum shearing stresses form angles of 45° with the planes of principal stresses.

Further, by making the triangle we get

\[
\cos 2\theta = \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} \\
\sin 2\theta = \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}
\]

Therefore by substituting the values of \(\cos 2\theta\) and \(\sin 2\theta\) we have

\[
\tau_e = \frac{1}{2} (\sigma_x - \sigma_y) \sin 2\theta - \tau_{xy} \cos 2\theta \\
= \frac{1}{2} \frac{(\sigma_x - \sigma_y)(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} - \frac{\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}
\]

\[
= -\frac{1}{2} \frac{(\sigma_y - \sigma_x)^2 + 4\tau_{xy}^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}
\]

\[
\tau_e = \frac{1}{2} \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}
\]

Because of root the difference in sign convention arises from the point of view of locating the planes on which shear stress act. From physical point of view these sign have no meaning.

The largest stress regard less of sign is always know as maximum shear stress.

**Principal plane inclination in terms of associated principal stress:**

\[
\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}
\]

We know that the equation

yields two values of \(q\) i.e. the inclination of the two principal planes on which the principal stresses \(s_1\) and \(s_2\) act. It is uncertain, however, which stress acts on which plane unless equation.
is used and observing which one of the two principal stresses is obtained.

Alternatively we can also find the answer to this problem in the following manner

Consider once again the equilibrium of a triangular block of material of unit depth, Assuming AC to be a principal plane on which principal stresses \( p \) acts, and the shear stress is zero.

Resolving the forces horizontally we get:

\[ x \cdot BC \cdot 1 + \ y \cdot AB \cdot 1 = \ p \cdot \cos \cdot AC \]

dividing the above equation through by BC we get

\[ \sigma_x + \frac{\tau_{xy}}{BC} = \frac{p \cdot \cos \theta \cdot AC}{BC} \]

or

\[ \sigma_x + \frac{\tau_{xy} \cdot \tan \theta}{BC} = \frac{p}{BC} \]

Thus

\[ \tan \theta = \frac{\sigma \ - \sigma_x}{\tau_{xy}} \]
GRAPHICAL SOLUTION – MOHR'S STRESS CIRCLE

The transformation equations for plane stress can be represented in a graphical form known as Mohr's circle. This graphical representation is very useful in depending the relationships between normal and shear stresses acting on any inclined plane at a point in a stresses body.

To draw a Mohr's stress circle consider a complex stress system as shown in the figure:

The above system represents a complete stress system for any condition of applied load in two dimensions.

The Mohr's stress circle is used to find out graphically the direct stress and shear stress on any plane inclined at to the plane on which \( x \) acts. The direction of here is taken in anticlockwise direction from the BC.

**STEPS:**

In order to do achieve the desired objective we proceed in the following manner:

(i) Label the Block ABCD.

(ii) Set up axes for the direct stress (as abscissa) and shear stress (as ordinate)

(iii) Plot the stresses on two adjacent faces e.g. AB and BC, using the following sign convention.

Direct stresses tension positive; compressive, negative
Shear stresses – tending to turn block clockwise, positive
– tending to turn block counter clockwise, negative

[ i.e shearing stresses are +ve when its movement about the centre of the element is clockwise ]
This gives two points on the graph which may then be labeled as $\overline{AB}$ and $\overline{BC}$ respectively to denote stresses on these planes.

(iv) Join $\overline{AB}$ and $\overline{BC}$.

(v) The point $P$ where this line cuts the $s$ axis is than the centre of Mohr’s stress circle and the line joining $\overline{AB}$ and $\overline{BC}$ is diameter. Therefore the circle can now be drawn.

Now every point on the circle then represents a state of stress on some plane through $C$.

Proof:
Consider any point Q on the circumference of the circle, such that PQ makes an angle 2 with BC, and drop a perpendicular from Q to meet the axis at N. Then OQ represents the resultant stress on the plane an angle to BC. Here we have assumed that \( x \) and \( y \).

Now let us find out the coordinates of point Q. These are ON and QN. From the figure drawn earlier

\[
ON = OP + PN
\]

\[
OP = OK + KP
\]

\[
OP = y + \frac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{y}{2} + \frac{y}{2} + \frac{x}{2} + \frac{y}{2} = \left( \frac{x + y}{2} \right)
\]

\[
PN = R \cos(2)
\]

hence ON = OP + PN

\[
= \left( \frac{x + y}{2} \right) + R \cos(2)
\]

\[
= \left( \frac{x + y}{2} \right) + R \cos^2 \theta + R \sin \theta \sin 2 \theta
\]

now make the substitutions for R cos and R sin.

\[
\text{Thus,}
\]

\[
ON = \frac{1}{2} (x + y) + \frac{1}{2} (x - y) \cos^2 + xy \sin^2 \quad (1)
\]

Similarly

\[
QM = R \sin(2)
\]

\[
= R \sin^2 \theta - R \cos^2 \theta
\]

Thus, substituting the values of R \cos and R \sin, we get

\[
QM = \frac{1}{2} (x - y) \sin^2 + xy \cos^2 \quad (2)
\]

If we examine the equation (1) and (2), we see that this is the same equation which we have already derived analytically.

Thus the co-ordinates of Q are the normal and shear stresses on the plane inclined at to BC in the original stress system.

**N.B:** Since angle \( \overline{BC} \) PQ is 2 on Mohr's circle and not it becomes obvious that angles are doubled on Mohr's circle. This is the only difference, however, as They are measured in the same direction and from the same plane in both figures.

Further points to be noted are:
(1) The direct stress is maximum when Q is at M and at this point obviously the shear stress is zero, hence by definition OM is the length representing the maximum principal stresses 1 and 2 gives the angle of the plane 1 from BC. Similar OL is the other principal stress and is represented by 2.

(2) The maximum shear stress is given by the highest point on the circle and is represented by the radius of the circle.

This follows that since shear stresses and complimentary shear stresses have the same value; therefore the centre of the circle will always lie on the s axis midway between x and y. [ since +xy & xy are shear stress & complimentary shear stress so they are same in magnitude but different in sign. ]

(3) From the above point the maximum shear stress i.e. the Radius of the Mohr's stress circle would be

\[
\frac{(\sigma_x - \sigma_y)}{2}
\]

While the direct stress on the plane of maximum shear must be mid – may between x and y i.e

\[
\frac{(\sigma_x + \sigma_y)}{2}
\]

(4) As already defined the principal planes are the planes on which the shear components are zero.

Therefore are conclude that on principal plane the shear stress is zero.

(5) Since the resultant of two stress at 90° can be found from the parallelogram of vectors as shown in the diagram. Thus, the resultant stress on the plane at q to BC is given by OQ on Mohr's Circle.
(6) The graphical method of solution for a complex stress problems using Mohr’s circle is a very powerful technique, since all the information relating to any plane within the stressed element is contained in the single construction. It thus, provides a convenient and rapid means of solution. Which is less prone to arithmetical errors and is highly recommended.

**Numericals:**

Let us discuss few representative problems dealing with complex state of stress to be solved either analytically or graphically.

**Q 1:** A circular bar 40 mm diameter carries an axial tensile load of 105 kN. What is the Value of shear stress on the planes on which the normal stress has a value of 50 MN/m² tensile.

**Solution:**

Tensile stress \( \sigma = \frac{F}{A} = \frac{105 \times 10^3}{\pi (0.02)^2} \)

\[ = 83.55 \text{ MN/m}^2 \]

Now the normal stress on an oblique plane is given by the relation

\[ \sigma = \sigma \sin^2 \theta \]

\[ 50 \times 10^6 = 83.55 \text{ MN/m}^2 \times 10^6\sin^2 \theta \]

\[ = 50^0.68^\circ \]

The shear stress on the oblique plane is then given by

\[ = \frac{1}{2} \sigma \sin 2\theta \]

\[ = \frac{1}{2} \times 83.55 \times 10^6 \times \sin 101.36 \]

\[ = 40.96 \text{ MN/m}^2 \]

Therefore the required shear stress is 40.96 MN/m²

**Q2:**

For a given loading conditions the state of stress in the wall of a cylinder is expressed as follows:
(a) 85 MN/m² tensile
(b) 25 MN/m² tensile at right angles to (a)
(c) Shear stresses of 60 MN/m² on the planes on which the stresses (a) and (b) act; the sheer couple acting on planes carrying the 25 MN/m² stress is clockwise in effect.

Calculate the principal stresses and the planes on which they act. What would be the effect on these results if owing to a change of loading (a) becomes compressive while stresses (b) and (c) remain unchanged

Solution:
The problem may be attempted both analytically as well as graphically. Let us first obtain the analytical solution

The principle stresses are given by the formula

\[ \sigma_1 \text{ and } \sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}} \]

\[ = \frac{1}{2}(85 + 25) \pm \frac{1}{2}\sqrt{(85 - 25)^2 + (4 \times 60^2)} \]

\[ = 55 \pm \frac{1}{2} \times 60 \sqrt{5} \approx 55 \pm 67 \]

\[ \Rightarrow \sigma_1 = 122 \text{ MN/m}^2 \]

\[ \sigma_2 = -12 \text{ MN/m}^2 \text{ (compressive)} \]

For finding out the planes on which the principle stresses act use the equation

\[ \tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \]

The solution of this equation will yield two values i.e they \( 1 \) and \( 2 \) giving \( 1 = 31^071' \) & \( 2 = 121^071' \)
(b) In this case only the loading (a) is changed i.e. its direction had been changed. While the other stresses remains unchanged hence now the block diagram becomes.

Again the principal stresses would be given by the equation.

\[ \sigma_1, \sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2} \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \]

\[ = \frac{1}{2}(-85 + 25) \pm \frac{1}{2} \sqrt{(-85 - 25)^2 + (4 \times 60^2)} \]

\[ = \frac{1}{2}(-60) \pm \frac{1}{2} \sqrt{(-85 - 25)^2 + (4 \times 60^2)} \]

\[ = -30 \pm \frac{1}{2} \sqrt{12100 + 14400} \]

\[ = -30 \pm 61.4 \]

\[ \sigma_1 = 51.4 \text{ MN/m}^2; \sigma_2 = -111.4 \text{ MN/m}^2 \]

Again for finding out the angles use the following equation.

\[ \tan 2\theta = \left( \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \right) \]

\[ = \frac{2 \times 60}{-85 - 25} = \frac{120}{-110} \]

\[ = -\frac{12}{11} \]

\[ 2\theta = \tan^{-1} \left( -\frac{12}{11} \right) \]

\[ \Rightarrow \theta = -23.74^0 \]

Thus, the two principle stresses acting on the two mutually perpendicular planes i.e principle planes may be depicted on the element as shown below:
So this is the direction of one principle plane & the principle stresses acting on this would be , when is acting normal to this plane, now the direction of other principal plane would be $90^\circ + \theta$ because the principal planes are the two mutually perpendicular plane, hence rotate the another plane $+ 90^\circ$ in the same direction to get the another plane, now complete the material element if $\theta$ is negative that means we are measuring the angles in the opposite direction to the reference plane BC.

Therefore the direction of other principal planes would be $\{+ 90\}$ since the angle is always less in magnitude then 90 hence the quantity ($+ 90$) would be positive therefore the Inclination of other plane with reference plane would be positive therefore if just complete the Block. It would appear as
If we just want to measure the angles from the reference plane, than rotate this block through $180^\circ$ so as to have the following appearance.

So whenever one of the angles comes negative to get the positive value, first add $90^\circ$ to the value and again add $90^\circ$ as in this case $\theta = 23^\circ74'$
so $\theta = 23^\circ74' + 90^\circ = 66^\circ26'$. Again adding $90^\circ$ also gives the direction of other principle planes.

i.e $\theta = 66^\circ26' + 90^\circ = 156^\circ26'$

This is how we can show the angular position of these planes clearly.

**GRAPHICAL SOLUTION:**

**Mohr's Circle solution:** The same solution can be obtained using the graphical solution i.e the Mohr's stress circle, for the first part, the block diagram becomes

Construct the graphical construction as per the steps given earlier.
Taking the measurements from the Mohr's stress circle, the various quantities computed are

1 = 120 MN/m² tensile
2 = 10 MN/m² compressive
1 = 34° counter clockwise from BC
2 = 34° + 90 = 124° counter clockwise from BC

Part Second: The required configuration i.e the block diagram for this case is shown along with the stress circle.

By taking the measurements, the various quantities computed are given as
1 = 56.5 MN/m² tensile
2 = 106 MN/m² compressive
1 = 66°15’ counter clockwise from BC
2 = 156°15’ counter clockwise from BC

**Salient points of Mohr’s stress circle:**
1. Complementary shear stresses (on planes 90° apart on the circle) are equal in magnitude.
2. The principal planes are orthogonal: points L and M are 180° apart on the circle (90° apart in material).
3. There are no shear stresses on principal planes: point L and M lie on normal stress axis.
4. The planes of maximum shear are 45° from the principal points D and E are 90°, measured round the circle from points L and M.
5. The maximum shear stresses are equal in magnitude and given by points D and E.
6. The normal stresses on the planes of maximum shear stress are equal i.e. points D and E both have normal stress co-ordinate which is equal to the two principal stresses.

As we know that the circle represents all possible states of normal and shear stress on any plane through a stresses point in a material. Further we have seen that the co-ordinates of the point ‘Q’ are seen to be the same as those derived from
equilibrium of the element. i.e. the normal and shear stress components on any plane passing through the point can be found using Mohr's circle. Worthy of note:

1. The sides AB and BC of the element ABCD, which are 90° apart, are represented on the circle by $\overline{AB}$ and $\overline{BC}$ and they are 180° apart.

2. It has been shown that Mohr's circle represents all possible states at a point. Thus, it can be seen at a point. Thus, it can be seen that two planes LP and PM, 180° apart on the diagram and therefore 90° apart in the material, on which shear stress is zero. These planes are termed as principal planes and normal stresses acting on them are known as principal stresses.

3. The maximum shear stress in an element is given by the top and bottom points of the circle i.e by points $J_1$ and $J_2$, Thus the maximum shear stress would be equal to the radius of i.e. $\max = \frac{1}{2}(1 + 2)$, the corresponding normal stress is obviously the distance $OP = \frac{1}{2}(x + y)$, Further it can also be seen that the planes on which the shear stress is maximum are situated 90° from the principal planes (on circle), and 45° in the material.

4. The minimum normal stress is just as important as the maximum. The algebraic minimum stress could have a magnitude greater than that of the maximum principal stress if the state of stress were such that the centre of the circle is to the left of origin.

i.e. if 

$$1 = 20 \text{ MN/m}^2 \text{ (say)}$$

$$2 = 80 \text{ MN/m}^2 \text{ (say)}$$

Then $\max = (\frac{1}{2} / 2) = 50 \text{ MN/m}^2$

5. Since the stresses on perpendicular faces of any element are given by the coordinates of two diametrically opposite points on the circle, thus, the sum of the two normal stresses for any and all orientations of the element is constant, i.e. Thus sum is an invariant for any particular state of stress.
Sum of the two normal stress components acting on mutually perpendicular planes at a point in a state of plane stress is not affected by the orientation of these planes.

This can be also understand from the circle Since AB and BC are diametrically opposite thus, whatever may be their orientation, they will always lie on the diametre or we can say that their sum won't change, it can also be seen from analytical relations

We know
\[
\sigma_n = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2} \cos 2\theta + \tau_{xy} \sin 2\theta
\]

on plane BC: \( n_1 = 0 \)

\( n_1 = x \)

on plane AB: \( n_2 = 270^\circ \)

\( n_2 = y \)

Thus \( n_1 + n_2 = x + y \)

6. If \( n_1 = n_2 \), the Mohr's stress circle degenerates into a point and no shearing stresses are developed on xy plane.

7. If \( x + y = 0 \), then the center of Mohr's circle coincides with the origin of co-ordinates.
Module 2

Lecture 6-7: Thin cylinder and thin spherical shells under internal pressure and numerical examples. Wire winding of thin cylinders. Numerical examples.

Cylindrical Vessel with Hemispherical Ends:

Let us now consider the vessel with hemispherical ends. The wall thickness of the cylindrical and hemispherical portion is different. While the internal diameter of both the portions is assumed to be equal.

Let the cylindrical vessel is subjected to an internal pressure $p$.

For the Cylindrical Portion

- Hoop circumferential stress $\sigma_{HC} = \frac{pd}{2t_2}$
- Longitudinal stress $\sigma_{LC} = \frac{pd}{4t_2}$

For the Hemispherical Ends

- Hoop circumferential strain $\varepsilon_2 = \frac{\sigma_{HC}}{E} - \nu \frac{\sigma_{LC}}{E} = \frac{pd}{4t_2E} [2 - \nu]$

or $\varepsilon_2 = \frac{pd}{4t_2E} [2 - \nu]$

For The Hemispherical Ends:
Because of the symmetry of the sphere the stresses set up owing to internal pressure will be two mutually perpendicular hoops or circumferential stresses of equal values. Again the radial stresses are neglected in comparison to the hoop stresses as with this cylinder having thickness to diameter less than 1:20.

Consider the equilibrium of the half-sphere

Force on half-sphere owing to internal pressure = pressure x projected Area = \( p \cdot \frac{d^2}{4} \)

Resisting force = \( \sigma_H \cdot \pi \cdot d \cdot t_2 \)

\[ \therefore \quad p \cdot \frac{\pi \cdot d^2}{4} = \sigma_H \cdot \pi \cdot d \cdot t_2 \]

\[ \Rightarrow \sigma_H \text{ (for sphere)} = \frac{pd}{4t_2} \]

Similarly the hoop strain = \( \frac{1}{E} \left[ \sigma_H - \nu \sigma_H \right] = \frac{\sigma_H}{E} \left[ 1 - \nu \right] = \frac{pd}{4t_2 E} \left[ 1 - \nu \right] \) or \( \varepsilon_{2S} = \frac{pd}{4t_2 E} \left[ 1 - \nu \right] \)

Fig - shown the (by way of dotted lines) the tendency, for the cylindrical portion and the spherical ends to expand by a different amount under the action of internal pressure. So owing to difference in stress, the two portions (i.e. cylindrical and spherical ends) expand by a different amount. This incompatibility of deformations causes a local bending and sheering stresses in the neighborhood of the joint. Since there must be physical continuity between the ends and the cylindrical portion, for this reason, properly curved ends must be used for pressure vessels.

Thus equating the two strains in order that there shall be no distortion of the junction

\[ \frac{pd}{4t_1 E} \left[ 2 - \nu \right] = \frac{pd}{4t_2 E} \left[ 1 - \nu \right] \] or \( \frac{t_2}{t_1} = \frac{1 - \nu}{2 - \nu} \)

But for general steel works \( \nu = 0.3 \), therefore, the thickness ratios becomes

\( t_2 / t_1 = 0.7 / 1.7 \) or

\( t_1 = 2.4t_2 \)

i.e. the thickness of the cylinder walls must be approximately 2.4 times that of the hemispheroid ends for no distortion of the junction to occur.
SUMMARY OF THE RESULTS: Let us summarise the derived results

(A) The stresses set up in the walls of a thin cylinder owing to an internal pressure $p$ are:

(i) Circumferential or loop stress

\[ \sigma_H = \frac{pd}{2t} \]

(ii) Longitudinal or axial stress

\[ \sigma_L = \frac{pd}{4t} \]

Where $d$ is the internal diameter and $t$ is the wall thickness of the cylinder.

then

Longitudinal strain \( \varepsilon_L = \frac{1}{E} \left[ \sigma_L - \nu \sigma_H \right] \)

Hoop strain \( \varepsilon_H = \frac{1}{E} \left[ \sigma_H - \nu \sigma_L \right] \)

(B) Change of internal volume of cylinder under pressure

\[ -\frac{pd}{4Et^{[5-4\nu]}}V \]

(C) For thin spheres circumferential or loop stress

\[ \sigma_H = \frac{bd}{4t} \]

Thin rotating ring or cylinder

Consider a thin ring or cylinder as shown in Fig below subjected to a radial internal pressure $p$ caused by the centrifugal effect of its own mass when rotating. The centrifugal effect on a unit length of the circumference is

\[ p = m \omega^2 r \]

Fig 19.1: Thin ring rotating with constant angular velocity
Here the radial pressure ‘p’ is acting per unit length and is caused by the centrifugal effect if its own mass when rotating.

Thus considering the equilibrium of half the ring shown in the figure,

\[ 2F = p \times 2r \text{ (assuming unit length), as } 2r \text{ is the projected area} \]

\[ F = pr \]

Where \( F \) is the hoop tension set up owing to rotation.

The cylinder wall is assumed to be so thin that the centrifugal effect can be assumed constant across the wall thickness.

\[ F = \text{mass x acceleration} = m \omega^2 r \times r \]

This tension is transmitted through the complete circumference and therefore is resisted by the complete cross-sectional area.

Hoop stress \( \sigma_{rr} = \frac{F}{A} = \frac{m \omega^2 r^2}{A} \)

Where \( A \) is the cross-sectional area of the ring.

Now with unit length assumed \( m/A \) is the mass of the material per unit volume, i.e. the density \( \rho \).

Hoop stress \( \sigma_{rr} = \rho \omega^2 r^2 \)